Newton Raphson Method

Notice: this material must <u>not</u> be used as a substitute for attending the lectures

0.1 Newton Raphson Method

The Newton Raphson method is for solving equations of the form $f(x) = 0$. We make an initial guess for the root we are trying to find, and we call this initial guess x_0. The sequence $x_0, x_1, x_2, x_3, \ldots$ generated in the manner described below should converge to the exact root.
To implement it analytically we need a formula for each approximation in terms of the previous one, i.e. we need x_{n+1} in terms of x_n.
The equation of the tangent line to the graph $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The tangent line intersects the x-axis when $y = 0$ and $x = x_1$, so

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

Solving this for x_1 gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and, more generally,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

You should **memorize** the above formula. Its application to solving equations of the form $f(x) = 0$, as we now demonstrate, is called the **Newton Raphson method**. It is guaranteed to converge if the initial guess x_0 is close enough, but it is hard to make a clear statement about what we mean by 'close enough' because this is highly problem specific. A sketch of the graph of $f(x)$ can help us decide on an appropriate initial guess x_0 for a particular problem.

0.2 Example

Let us solve $x^3 - x - 1 = 0$ for x.
In this case $f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$. So the recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 1)}{3x_n^2 - 1}$$

Need to decide on an appropriate initial guess x_0 for this problem. A rough graph can help. Note that $f(1) = -1 < 0$ and $f(2) = 5 > 0$. Therefore, a root of $f(x) = 0$ must exist between 1 and 2. Let us take $x_0 = 1$ as our initial guess. Then

$$x_1 = x_0 - \frac{(x_0^3 - x_0 - 1)}{3x_0^2 - 1}$$

and with $x_0 = 1$ we get $x_1 = 1.5$.
Now

$$x_2 = x_1 - \frac{(x_1^3 - x_1 - 1)}{3x_1^2 - 1}$$

and with $x_1 = 1.5$ we get $x_2 = 1.34783$. For the next stage,

$$x_3 = x_2 - \frac{(x_2^3 - x_2 - 1)}{3x_2^2 - 1}$$

and with the value just found for x_2, we find $x_3 = 1.32520$.
Carrying on, we find that $x_4 = 1.32472$, $x_5 = 1.32472$, etc. We can stop when the digits stop changing to the required degree of accuracy. We conclude that the root is 1.32472 to 5 decimal places.

0.3 Example

Let us solve $\cos x = 2x$ to 5 decimal places.
This is equivalent to solving $f(x) = 0$ where $f(x) = \cos x - 2x$. [**NB: make sure your calculator is in radian mode**]. The recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(\cos x_n - 2x_n)}{(-\sin x_n - 2)}$$

With an initial guess of $x_0 = 0.5$, we obtain:

$$\begin{aligned} x_0 &= 0.5 \\ x_1 &= 0.45063 \\ x_2 &= 0.45018 \\ x_3 &= 0.45018 \\ &\vdots \end{aligned}$$

with no further changes in the digits, to five decimal places. Therefore, to this degree of accuracy, the root is $x = 0.45018$.

0.4 Possible problems with the method

The Newton-Raphson method works most of the time if your initial guess is good enough. Occasionally it fails but sometimes you can make it work by changing the initial guess. Let's try to solve $x = \tan x$ for x. In other words, we solve $f(x) = 0$ where $f(x) = x - \tan x$. The recursion formula (1) becomes

$$x_{n+1} = x_n - \frac{(x_n - \tan x_n)}{1 - \sec^2 x_n}$$

Let's try an initial guess of $x_0 = 4$. With this initial guess we find that $x_1 = 6.12016$, $x_2 = 238.40428$, $x_3 = 1957.26490$, etc. Clearly these numbers are not converging.
We need a new initial guess. Let's try $x_0 = 4.6$. Then we find $x_1 = 4.54573$, $x_2 = 4.50615$, $x_3 = 4.49417$, $x_4 = 4.49341$, $x_5 = 4.49341$, etc. A couple of further iterations will confirm that the digits are no longer changing to 5 decimal places. As a result, we conclude that a root of $x = \tan x$ is $x = 4.49341$ to 5 decimal places.

Solutions to Problems on the Newton-Raphson Method

These solutions are not as brief as they should be: it takes work to be brief. There will, almost inevitably, be some numerical errors. Please inform me of them at `adler@math.ubc.ca`. We will be excessively casual in our notation. For example, $x_3 = 3.141592654$ will mean that the calculator gave this result. It does not imply that x_3 is exactly equal to 3.141592654.

We should always treat at least the final digit of a calculator answer with some skepticism. Indeed different calculators can give (mildly) different answers. In applied work, we need to pay heed to the fact that the standard tools, such as calculators and computer programs, work only to limited precision. In a complex calculation, minor inaccuracies may result in a significant error.

1. Use the Newton-Raphson method, with 3 as starting point, to find a fraction that is within 10^{-8} of $\sqrt{10}$. Show (without using the square root button) that your answer is indeed within 10^{-8} of the truth.

   Solution: The number $\sqrt{10}$ is the unique positive solution of the equation $f(x) = 0$ where $f(x) = x^2 - 10$. We use the Newton Method to approximate a solution of this equation.

   Let x_0 be our initial estimate of the root, and let x_n be the n-th improved estimate. Note that $f'(x) = 2x$. The Newton Method recurrence is therefore

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 10}{2x_n}.$$

   To make the expression on the right more beautiful, and calculations easier, it is useful to manipulate it a bit. We get

$$x_{n+1} = x_n - \frac{x_n}{2} + \frac{10}{2x_n} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right).$$

Compute, starting with $x_0 = 3$. Then $x_1 = (1/2)(x_0 + 10/x_0) = (1/2)(3 + 10/3) = 19/6$. And $x_2 = (1/2)(19/6 + 60/19) = 721/228$. We could go on calculating with fractions—and there is interesting mathematics involved—but from here on we switch to the calculator.

If we allow the $=$ sign to be used sloppily, we get $x_1 = 3.166666667$. Then $x_2 = (1/2)(x_1 + 10/x_1) = 3.162280702$, and $x_3 = 3.16227766$, and $x_4 = 3.16227766$.

The calculator says that $x_3 = x_4$ to 8 decimal places. We can therefore dare hope that 3.16227766 is close enough. One way of checking is to let $a = 3.16227765$ and $b = 3.16227767$. A quick calculation shows—if the squaring button can be trusted, and it is one of the ones that can be—that $f(a) < 0$ while $f(b) > 0$.

Thus the function $f(x)$ changes sign as x goes from a to b. It follows by the Intermediate Value Theorem that $f(x) = 0$ has a solution (namely $\sqrt{10}$) between a and b. Since $\sqrt{10}$ lies in the interval (a, b), and the distance from 3.16227766 to either a or b is 10^{-8}, it follows that the distance from 3.16227766 to $\sqrt{10}$ is less than 10^{-8}.

2. Let $f(x) = x^2 - a$. Show that the Newton Method leads to the recurrence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Heron of Alexandria (60 CE?) used a pre-algebra version of the above recurrence. It is still at the heart of computer algorithms for finding square roots.

Solution: We have $f(x) = 2x$. The Newton Method therefore leads to the recurrence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n}.$$

Bring the expression on the right hand side to the common denominator $2x_n$. We get

$$x_{n+1} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

3. Newton's equation $y^3 - 2y - 5 = 0$ has a root near $y = 2$. Starting with $y_0 = 2$, compute y_1, y_2, and y_3, the next three Newton-Raphson estimates for the root.

Solution: Let $f(y) = y^3 - 2y - 5$. Then $f'(y) = 3y^2 - 2$, and the Newton Method produces the recurrence

$$y_{n+1} = y_n - \frac{y_n^3 - 2y_n - 5}{3y_n^2 - 2} = \frac{2y_n^3 + 5}{3y_n^2 - 2}$$

(there was no good case for simplification here). Start with the estimate $y_0 = 2$. Then $y_1 = 21/10 = 2.1$. It follows that (to calculator accuracy) $y_2 = 2.094568121$ and $y_3 = 2.094551482$. These are almost the numbers that Newton obtained (see the notes). But Newton in effect used a rounded version of y_2, namely 2.0946.

4. Find all solutions of $e^{2x} = x + 6$, correct to 4 decimal places; use the Newton Method.

   Solution: Let $f(x) = e^{2x} - x - 6$. We want to find where $f(x) = 0$. Note that $f'(x) = 2e^{2x} - 1$, so the Newton Method iteration is

$$x_{n+1} = x_n - \frac{e^{2x_n} - x_n - 6}{2e^{2x_n} - 1} = \frac{(2x_n - 1)e^{2x_n} + 6}{2e^{2x_n} - 1}.$$

   We need to choose an initial estimate x_0. This can be done in various ways. We can (if we are rich) use a graphing calculator or a graphing program to graph $y = f(x)$ and eyeball where the graph crosses the x-axis. Or else, if (like the writer) we are poor, we can play around with a cheap calculator, a slide rule, an abacus, or scrap paper and a dull pencil.

   It is easy to verify that $f(1)$ is about 0.389, and that $f(0.95)$ is about -0.2641, so by the Intermediate Value Theorem there is a root between 0.95 and 1. And since $f(0.95)$ is closer to 0 than is $f(1)$, maybe the root is closer to 0.95 than to 1. Let's make the initial estimate $x_0 = 0.97$.

   The calculator then gives $x_1 = 0.970870836$, and then $x_2 = 0.97087002$. Since these two agree to 5 decimal places, we can perhaps conclude with some (but not complete) assurance that the root, to 4 decimal places, is 0.9709. If we want greater assurance, we can compute $f(0.97085)$ and $f(0.97095)$ and hope for a sign change, which shows that there is a root between 0.97085 and 0.97095. There is indeed such a sign change: $f(0.97085)$ is about -2.6×10^{-4} while $f(0.97095)$ is about 10^{-3}.

   But the problem asked for *all* the solutions. Are there any others?

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$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and, more generally,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

You should **memorize** the above formula. Its application to solving equations of the form $f(x) = 0$, as we now demonstrate, is called the **Newton Raphson method**. It is guaranteed to converge if the initial guess x_0 is close enough, but it is hard to make a clear statement about what we mean by 'close enough' because this is highly problem specific. A sketch of the graph of $f(x)$ can help us decide on an appropriate initial guess x_0 for a particular problem.

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$$x_1 = x_0 - \frac{(x_0^3 - x_0 - 1)}{3x_0^2 - 1}$$

and with $x_0 = 1$ we get $x_1 = 1.5$.
Now

$$x_2 = x_1 - \frac{(x_1^3 - x_1 - 1)}{3x_1^2 - 1}$$

and with $x_1 = 1.5$ we get $x_2 = 1.34783$. For the next stage,

$$x_3 = x_2 - \frac{(x_2^3 - x_2 - 1)}{3x_2^2 - 1}$$

and with the value just found for x_2, we find $x_3 = 1.32520$.
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Topic 10.1: Bisection Method (Examples)

Introduction Notes Theory HOWTO Examples Engineering Error Questions Matlab Maple

Example 1

Consider finding the root of f(x) = x^2 - 3. Let ε_{step} = 0.01, ε_{abs} = 0.01 and start with the interval [1, 2].

Table 1. Bisection method applied to f(x) = x^2 - 3.

a	*b*	f(*a*)	f(*b*)	*c* = (a + b)/2	f(*c*)	Update	new b − a
1.0	2.0	-2.0	1.0	1.5	-0.75	a = c	0.5
1.5	2.0	-0.75	1.0	1.75	0.062	b = c	0.25
1.5	1.75	-0.75	0.0625	1.625	-0.359	a = c	0.125
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	a = c	0.0625
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	a = c	0.0313
1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	b = c	0.0156
1.71988/td>	1.7344	-0.0457	0.0081	1.7266	-0.0189	a = c	0.0078

Thus, with the seventh iteration, we note that the final interval, [1.7266, 1.7344], has a width less than 0.01 and |f(1.7344)| < 0.01, and therefore we chose b = 1.7344 to be our approximation of the root.

Example 2

Consider finding the root of f(x) = e^{-x}(3.2 sin(x) - 0.5 cos(x)) on the interval [3, 4], this time with ε_{step} = 0.001, ε_{abs} = 0.001.

Table 1. Bisection method applied to f(x) = e^{-x}(3.2 sin(x) - 0.5 cos(x)).

a	*b*	f(*a*)	f(*b*)	*c* = (a + b)/2	f(*c*)	Update	new b − a
3.0	4.0	0.047127	-0.038372	3.5	-0.019757	b = c	0.5
3.0	3.5	0.047127	-0.019757	3.25	0.0058479	a = c	0.25
3.25	3.5	0.0058479	-0.019757	3.375	-0.0086808	b = c	0.125
3.25	3.375	0.0058479	-0.0086808	3.3125	-0.0018773	b = c	0.0625

3.25	3.3125	0.0058479	-0.0018773	3.2812	0.0018739	a = c	0.0313
3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.000024791	b = c	0.0156
3.2812	3.2968	0.0018739	-0.000024791	3.289	0.00091736	a = c	0.0078
3.289	3.2968	0.00091736	-0.000024791	3.2929	0.00044352	a = c	0.0039
3.2929	3.2968	0.00044352	-0.000024791	3.2948	0.00021466	a = c	0.002
3.2948	3.2968	0.00021466	-0.000024791	3.2958	0.000094077	a = c	0.001
3.2958	3.2968	0.000094077	-0.000024791	3.2963	0.000034799	a = c	0.0005

Thus, after the 11th iteration, we note that the final interval, [3.2958, 3.2968] has a width less than 0.001 and |f(3.2968)| < 0.001 and therefore we chose b = 3.2968 to be our approximation of the root.

Example 3

Apply the bisection method to f(*x*) = sin(*x*) starting with [1, 99], $\varepsilon_{step} = \varepsilon_{abs}$ = 0.00001, and comment.

After 24 iterations, we have the interval [40.84070158, 40.84070742] and sin(40.84070158) ≈ 0.0000028967. Note however that sin(*x*) has 31 roots on the interval [1, 99], however the bisection method neither suggests that more roots exist nor gives any suggestion as to where they may be.

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http://www.ece.uwaterloo.ca/~ece104/

Question: Determine the root of the given equation $x^2-3 = 0$ for $x \in [1, 2]$

Solution:

Given: $x^2-3 = 0$

Let $f(x) = x^2-3$

Now, find the value of f(x) at a= 1 and b=2.

$f(x=1) = 1^2-3 = 1 – 3 = -2 < 0$

$f(x=2) = 2^2-3 = 4 – 3 = 1 > 0$

The given function is continuous, and the root lies in the interval [1, 2].

Let "t" be the midpoint of the interval.

I.e., t = (1+2)/2

t =3 / 2

t = 1.5

Therefore, the value of the function at "t" is

$f(t) =f(1.5) = (1.5)^2-3 = 2.25 – 3 = -0.75 < 0$

f(t) is negative, so b is replaced with t= 1.5 for the next iterations.

The iterations for the given functions are:

Iterations	a	b	t	f(a)	f(b)	f(t)
1	1	2	1.5	-2	1	-0.75
2	1.5	2	1.75	-0.75	1	0.062
3	1.5	1.75	1.625	-0.75	0.0625	-0.359
4	1.625	1.75	1.6875	-0.3594	0.0625	-0.1523
5	1.6875	1.75	1.7188	-01523	0.0625	-0.0457
6	1.7188	1.75	1.7344	-0.0457	0.0625	0.0081
7	1.7188	1.7344	1.7266	-0.0457	0.0081	-0.0189

So, at the seventh iteration, we get the final interval [1.7266, 1.7344]

Hence, 1.7344 is the approximated solution.

Related Links	
Calculus (https://byjus.com/maths/calculus/)	Limits and Derivatives (https://byjus.com/maths/limits-and-derivatives/)
Nature of Roots Quadratic (https://byjus.com/maths/nature-of-roots-quadratic/)	Domain Codomain Range Functions (https://byjus.com/maths/domain-codomain-range-functions/)

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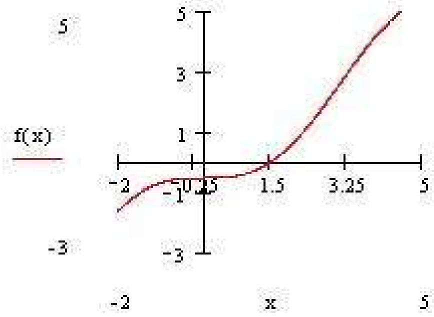
Quadrilaterals Worksheet (https://byjus.com/maths/quadrilateral-worksheet/)	Euclid's Axioms (https://byjus.com/maths/euclidean-geometry/)

5. Find the root of **x - sin(x) - (1/2)= 0**

The graph of this equation is given in the figure.

Let a = 1 and b = 2



Iteration No.	a	b	c	f(a) * f(c)
1	1	2	1.5	$-8.554*10^{-4}$ (-ve)
2	1	1.5	1.25	0.068 (+ve)
3	1.25	1.5	1.375	0.021 (+ve)
4	1.375	1.5	1.437	$5.679*10^{-3}$ (+ve)
5	1.437	1.5	1.469	$1.42*10^{-3}$ (+ve)
6	1.469	1.5	1.485	$3.042*10^{-4}$ (+ve)
7	1.485	1.5	1.493	$5.023*10^{-5}$ (+ve)
8	1.493	1.5	**1.497**	$2.947*10^{-6}$ (+ve)

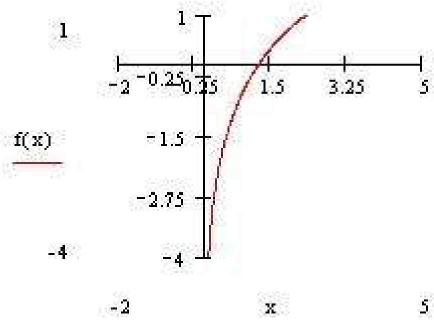
So one of the roots of x-sin(x)-(1/2) = 0 is approximately 1.497.

BACK

6. Find the root of **exp(-x)=3log(x)**

The graph of this equation is given in the figure.

Let a = 0.5 and b = 1.5



Iteration No.	a	b	c	f(a) * f(c)
1	0.5	1.5	1	0.555 (+ve)
2	1.00	1.5	1.25	$-1.554*10^{-3}$ (-ve)
3	1.00	1.25	1.125	0.063 (+ve)
4	1.125	1.25	**1.187**	0.014 (+ve)

So one of the roots of exp(-x)=3log(x) is approximately 1.187.

BACK

Problems to Work-Out:

7. Find the root of x * cos[(x)/ (x-2)] = 0 [*Graph*]

8. [*Graph*]

Find the root of $x^2 = (\exp(-2x) - 1) / x$

9. **Find the root of $\exp(x^2-1)+10\sin 2x-5 = 0$** [*Graph*]

10. **Find the root of $\exp(x)-3x^2=0$** [*Graph*]

11. **Find the root of $\tan(x)-x-1 = 0$** [*Graph*]

12. **Find the root of $\sin(2x)-\exp(x-1) = 0$** [*Graph*]

BACK

FIXED POINT ITERATION METHOD

Fixed point : A point, say, **s** is called a fixed point if it satisfies the equation **$x = g(x)$**.

Fixed point Iteration : The transcendental equation **$f(x) = 0$** can be converted algebraically into the form **$x = g(x)$** and then using the iterative scheme with the recursive relation

$$x_{i+1}= g(x_i), \qquad i = 0, 1, 2, \ldots,$$

with some initial guess x_0 is called the fixed point iterative scheme.

Algorithm - Fixed Point Iteration Scheme

Given an equation $f(x) = 0$
Convert $f(x) = 0$ into the form $x = g(x)$
Let the initial guess be x_0
Do
    $x_{i+1}= g(x_i)$
while (none of the convergence criterion C1 or C2 is met)

- C1. Fixing apriori the total number of iterations **N** .
- C2. By testing the condition $| x_{i+1} - g(x_i) |$ (where **i** is the iteration number) less than some tolerance limit, say epsilon, fixed apriori.

Numerical Example :

Find a root of $x^4-x-10 = 0$ [*Graph*]
Consider **$g1(x) = 10 / (x^3-1)$** and the fixed point iterative scheme $x_{i+1}=10 / (x_i^3 -1)$, $i = 0, 1, 2, \ldots$ let the initial guess x_0 be **2.0**

i	0	1	2	3	4	5	6	7	8
x_i	2	1.429	5.214	0.071	-10.004	-9.978E-3	-10	-9.99E-3	-10

So the iterative process with **g1** gone into an infinite loop without converging.

Consider another function **$g2(x) = (x + 10)^{1/4}$** and the fixed point iterative scheme
$x_{i+1}= (x_i + 10)^{1/4}$, $i = 0, 1, 2, \ldots$

let the initial guess x_0 be **1.0, 2.0 and 4.0**

i	0	1	2	3	4	5	6

x_i	1.0	1.82116	1.85424	1.85553	1.85558	**1.85558**	
x_i	2.0	1.861	1.8558	1.85559	1.85558	**1.85558**	
x_i	4.0	1.93434	1.85866	1.8557	1.85559	1.85558	**1.85558**

That is for **g2** the iterative process is converging to **1.85558 with** any initial guess.

Consider $\mathbf{g3(x) = (x+10)^{1/2}/x}$ and the fixed point iterative scheme

$$x_{i+1} = (x_i + 10)^{1/2} / x_i, \quad i = 0, 1, 2, \ldots$$

let the initial guess x_0 be **1.8,**

i	0	1	2	3	4	5	6	...	98
x_i	1.8	1.9084	1.80825	1.90035	1.81529	1,89355	1.82129	...	**1.8555**

That is for g3 with any initial guess the iterative process is converging but very slowly to

Geometric interpretation of convergence with g1, g2 and g3

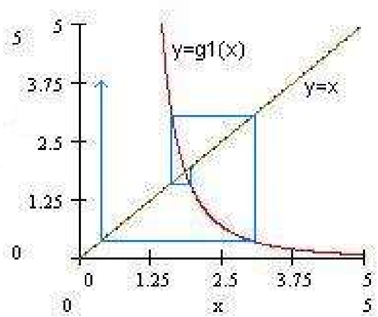


Fig g1

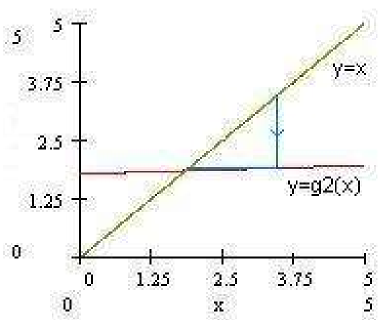


Fig g2

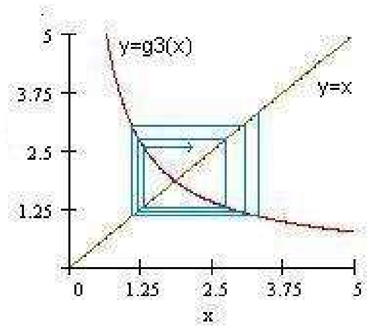


Fig g3

The graphs Figures Fig g1, Fig g2 and Fig g3 demonstrates the Fixed point Iterative Scheme with g1, g2 and g3 respectively for some initial approximations. It's clear from the

- Fig g1, the iterative process does not converge for any initial approximation.
- Fig g2, the iterative process converges very quickly to the root which is the intersection point of y = x and y = g2(x) as shown in the figure.
- Fig g3, the iterative process converges but very slowly.

Example 2 **:**The equation $\mathbf{x^4 + x} = \in$, where $\in$ is a small number , has a root which is close to $\in$. Computation of this root is done by the expression $\xi = \in - \in^4 + 4\in^7$ Then find an iterative formula of the form $\mathbf{x_{n+1} = g(x_n)}$, if we start with $x_0 = \mathbf{0}$ for the computation then show that we get the expression given above as a solution. Also find the error in the approximation in the interval **[0, 0.2]** .

Proof

Given $\mathbf{x^4 + x} = \in$

$\mathbf{x(x^3 + 1)} = \in$

$x = \in/(1 + x^3)$ or $x_i = \in/(1 + x_i^{\,3})$ $i = 0, 1, 2, \ldots$

$x_0 = 0$

$x_1 = \in$

$x_2 = \in/\ (1 + \in_i^{\,3}) = \in(1 + \in_i^{\,3})^{-1}$
$= \in(1 - \in^3 + \in^6 + \ldots)$
$= \in - \in^4 + \in^7 + \ldots$

$x_3 = \in/(\ 1 + (\in - \in^4 + \in^7)^3) = \in[1 + (\in - \in^4 + \in^7)^{-3}] = \in - \in^4 + 4\in^7$

Now taking $\xi = \in - \in^4 + 4\in^7$

error $= \xi^4 + \xi - \in$
$= (\in - \in^4 + 4\in^7)^4 + (\in - \in^4 + 4\in^7) - \in$
$= 22\in^{10}$ + **higher order power of** $\in$

Condition for Convergence :

If **g(x)** and **g'(x)** are continuous on an interval **J** about their root **s** of the equation **x = g(x)**, and if **|g'(x)|<1** for all **x** in the interval **J** then the fixed point iterative process $x_{i+1}=g(x_i)$, **i = 0, 1, 2, . . .**, will converge to the root **x = s** for any initial approximation x_0 belongs to the interval **J** .

[*Proof*]

Worked out problems

Exapmple 1	Find a root of cos(x) - x * exp(x) = 0	**Solution**
Exapmple 2	Find a root of x^4-x-10 = 0	**Solution**
Exapmple 3	Find a root of x-exp(-x) = 0	**Solution**
Exapmple 4	Find a root of exp(-x) * (x^2-5x+2) + 1= 0	**Solution**
Exapmple 5	Find a root of x-sin(x)-(1/2)= 0	**Solution**
Exapmple 6	Find a root of exp(-x) = 3log(x)	**Solution**
Problems to workout		

Work out with the **Fixed Point Iteration** method here

Note :Few examples of how to enter equations are given below . . . (i) exp[-x]*(x^2+5x+2)+1 (ii) x^4-x-10 (iii) x-sin[x]-(1/2) (iv) exp[(-x+2-1-2+1)]*(x^2+5x+2)+1 (v) (x+10) ^ (1/4)

BACK NEXT

Solution of Transcendental Equations | Solution of Linear System of Algebraic Equations | Interpolation & Curve Fitting
Numerical Differentiation & Integration | Numerical Solution of Ordinary Differential Equations
Numerical Solution of Partial Differential Equations

search

System of nonlinear equation - Newton - Raphson method:

Example: Solve the following nonlinear equations using Newton - method.

start at $x=1$ and $y=1$

$$4x^2 - y^3 + 28 = 0$$

$$3x^3 + 4y^2 - 145 = 0$$

Solution: The formula of the solution is

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left[J(x_0, y_0) \right]^{-1} \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix}$$

Let $f_1(x,y) = 4x^2 - y^3 + 28 = 0$

$f_2(x,y) = 3x^3 + 4y^2 - 145 = 0$

Now to comput the Jacobian determinent as follows:

$$J(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$f_1 = 4x^2 - y^3 + 28$$

$$\frac{\partial f_1}{\partial x} = 8x \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -3y^2$$

$$f_2 = 3x^3 + 4y^2 - 145$$

$$\frac{\partial f_2}{\partial x} = 9x^2 \quad \text{and} \quad \frac{\partial f_2}{\partial y} = 8y$$

$$J(x,y) = \begin{bmatrix} 8x & -3y^2 \\ 9x^2 & 8y \end{bmatrix}$$

$$[J(x,y)]^{-1} = \frac{1}{\det J} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{64xy + 27y^2x^2} \begin{bmatrix} 8y & 3y^2 \\ -9x^2 & 8x \end{bmatrix}$$

(21)

$$\left[J(x_0, y_0)\right]^{-1} = \frac{1}{91}\begin{bmatrix} 8 & 3 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 0.08791 & 0.0329 \\ -0.0989 & 0.08791 \end{bmatrix}$$

$$f_1(x_0, y_0) = 4(1)^2 - (1)^3 + 28 = 31$$

$$f_2(x_0, y_0) = 3(1)^3 + 4(1)^2 - 145 = -138$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left[J(x, y)\right]^{-1} \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.08791 & 0.03296 \\ -0.0989 & 0.0879 \end{bmatrix} \begin{bmatrix} 31 \\ -138 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2.7252 & -4.549 \\ -3.065 & -12.1315 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1.8238 \\ -15.1975 \end{bmatrix} = \begin{bmatrix} 2.8238 \\ 16.1975 \end{bmatrix}$$

r'e

_______: Solve the following non linear system using Newton method ?

$$\sin(3x) + \cos y = 0$$

$$\cos 5x + \sin y = 0$$

where $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution

_______: Let $f_1 = \sin 3x + \cos y$

$f_2 = \cos 5x + \sin y$

$$J(x,y) = \begin{bmatrix} 3\cos 3x & -\sin y \\ -5\sin 5x & \cos y \end{bmatrix}$$

$$[J(x,y)]^{-1} = \frac{1}{3\cos y\cos 3x - 5\sin 5x \sin y} \begin{bmatrix} \cos y & \sin y \\ 5\sin 5x & 3\cos 3x \end{bmatrix}$$

$$= \frac{1}{2.4298} \begin{bmatrix} 0.5403 & 0.8414 \\ -4.7946 & -2.9699 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2223 & 0.3462 \\ -1.9732 & -1.22228 \end{bmatrix}$$

$f_1(x_0, y_0) = 0.6814$ and $f_2(x_0, y_0) = 1.1251$

$$\begin{bmatrix} 0.2223 & 0.3462 \\ -1.9732 & -1.22228 \end{bmatrix} \begin{bmatrix} 0.6814 \\ 1.125 \end{bmatrix} =$$

$$\begin{bmatrix} 0.5409 \\ -2.7196 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.5409 \\ -2.7196 \end{bmatrix}$$

$$= \begin{matrix} 0.4591 \\ 3.7196 \end{matrix}$$

H.W

Solve the following non linear equations using Newton method.

$$\begin{bmatrix} 6.6763 \\ 10.735 \end{bmatrix}$$

$$x^2 - xy + 20 = 0$$

$$y^2 - 2xy + 10 = 0$$

start at $x = 6$ and $y = 10$

Gaussin Elimination method

The aim of this method is to convert the coefficients matrix into an upper triangular matrix using forward elimination and the using back substitution to find x_i, as in the following examples:

Example: Find the solution for the follo example, using Gaussin elimination method

$$x + y + z = 6$$
$$2x + y - z = 1$$
$$-x + 2y + 2z = 9$$

Solution:

step 1 write the system in matrix form →

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 2 & 9 \end{array}\right]$$

(26) (24)

Step 2: We notice that the element a_{21} is the largest element in the first column of A, so we will re arrange our matrix so that the largest element of the first column will be our a_{11} $\Rightarrow$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 6 \\ -1 & 2 & 2 & 9 \end{array}\right]$$

- this process called partial pivoting.

step 3 we must have $a_{i1} = 0$ $(i = 2, 3)$ using

$$R_i = R_i - \left(\frac{a_{i1}}{a_{11}}\right) R_1$$

أي السطر الجديد = السطر القديم − (القيمة المراد تصفيرها / القيمة التي تحتل pivot تساوي (1)) × سطر الـ pivot

as follows

$$R_2 = 1 - \left(\tfrac{1}{2}\right)(2) = 0$$
$$= 1 - \left(\tfrac{1}{2}\right)(1) = \tfrac{1}{2}$$
$$= 1 - \left(\tfrac{1}{2}\right)(-1) = \tfrac{3}{2}$$
$$= 6 - \left(\tfrac{1}{2}\right)(1) = \tfrac{11}{2}$$

$R_3 = -1 - \left(\frac{-1}{2}\right)(2) = 0$

$= 2 - \left(\frac{-1}{2}\right)(1) = \frac{5}{2}$

$= 2 - \left(\frac{-1}{2}\right)(-1) = \frac{3}{2}$

$= 9 - \left(\frac{-1}{2}\right)(1) = \frac{19}{2}$

and we put the privious results in matrix form $\Longrightarrow$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 1/2 & 3/2 & 11/2 \\ 0 & 5/2 & 3/2 & 19/2 \end{array}\right]$$

step 4: after noticing that a_{22} is not the largest element in the 2nd column, we use partial pivoting again to get

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 5/2 & 3/2 & 19/2 \\ 0 & 1/2 & 3/2 & 11/2 \end{array}\right]$$

and then we use the formula:

$R_i = R_i - \left(\frac{a_{i2}}{a_{22}}\right) R_2 \quad , \quad i = 3 \Longrightarrow$

$R_3 : \frac{1}{2} - \left(\frac{1}{2} \cdot \frac{2}{5} \right) \frac{5}{2} = 0$

$= 3/2 - \left(\frac{1}{5} \right) \cdot \frac{3}{2} = \frac{3}{2} - \frac{3}{10} = \frac{6}{5}$

$= \frac{11}{2} - \left(\frac{1}{5} \right) \frac{19}{2} = \frac{11}{2} - \frac{19}{10} = \frac{55-19}{10} = \frac{18}{5}$

and put the results in matrix form →

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 5/2 & 3/2 & 19/2 \\ 0 & 0 & 6/5 & 18/5 \end{bmatrix}$$

Step 5: As we get an upper tringular matrix, we will now use the back substitution to find the values of x, y & z as follows

$z = \frac{18}{5} \cdot \frac{5}{6} = \boxed{3}$

$y = \left[\frac{19}{2} - \frac{3}{2} x_3 \right] \frac{2}{5} = \left[\frac{19}{2} - \frac{3}{2} (3) \right] \cdot \frac{2}{5} = \boxed{2}$

$x = [1 - x_2 + x_3] \frac{1}{2}$

$= [1 - 2 + 3] \frac{1}{2} = \boxed{1}$

Example: Using Gaussina elimination, find the values of x, y, z of the following system:

$$2x + y - z = 2$$
$$x - y + z = 7$$
$$2x = 4 - 2y - z$$

Solution

step 1: writ the system in matrix form

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 7 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

step 2: no need for partial pivoting $\Longrightarrow$

$$R_2 = 1 - (\tfrac{1}{2})(2) = 0$$
$$= -1 - (\tfrac{1}{2})(1) = \frac{-3}{2}$$
$$= 1 - (\tfrac{1}{2})(-1) = \frac{3}{2}$$
$$= 7 - (\tfrac{1}{2})(2) = 6$$

$R_3 = 2 - \left(\frac{2}{2}\right) 2 = 0$

$= 2 - (1)(1) = 1$

$= 1 - (1)(-1) = 2$

$= 4 - (1)(2) = 2$

~~step~~ put the results in matrix form

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & -3/2 & 3/2 & 6 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

step 3

$R_3 = 1 - \frac{1}{(-3/2)}(-3/2) = 0$

$= 2 + \left(\frac{2}{3}\right)\left(\frac{3}{2}\right) = 3$

$= 2 + \left(\frac{2}{3}\right) \times 6 = 6$

put the results in matrix form

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 0 & -3/2 & 3/2 & 6 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Step 4: Use back substitution to find th values of x, y + z as follows

$$3x_3 = 6 \Rightarrow \boxed{x_3 = 2}$$

$$x_2 = \left[6 - \frac{3}{2}x_3\right]\frac{-2}{3}$$
$$= \left[6 - \frac{3}{2}\cdot 2\right]\frac{-2}{3} = \boxed{-2}$$

$$x_1 = \left[2 - x_2 + x_3\right]\frac{1}{2}$$
$$= \left[2 + 2 + 2\right]\frac{1}{2} = \boxed{3}$$

H.W Use Gaussian Elimination to solve the following system:

$$x + 2y + z = 8$$
$$3x + 4y + 2z = 17$$
$$6y - 20 = -5x - z$$

Answer $x = 1$
$y = 2$
$z = 3$

(KM) N32

Numerical Methods for Solving Differential Equation

An intial value problem cosists of a differential equation and a condition, the solution must satisfy that condition The intial problem considered here is the form

$$y' = f(x, y) \quad , y(x_0) = y_0$$

① Euler Method

This method is the easiest method to solve the differential Equation. Its formula reads

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where $x_{n+1} = x_n + h$

and h is the step size.

Example: Use Euler's method to obtain an approximate solution of the following differential equation

$$y' = x^2 + 4x - \frac{1}{2}y$$

at $x = .25$, $y(0) = 4$ and $h = .05$

where

N 33

Solution:

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad y' = f(x, y)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 4 + (.05)\left[(0) + 4(0) - \tfrac{1}{2}(4) \right] = \boxed{3.9}$$

$$x_{n+1} = x_n + h \Rightarrow x_1 = x_0 + .05 = .05$$

$$y_2 = y_1 + h\left[x_1^2 + 4x_1 - \tfrac{1}{2} y_1 \right]$$

$$= 3.9 + (.05)\left[(.05)^2 + 4(.05) - \tfrac{1}{2}(3$$

$$= \boxed{3.8126}$$

$$x_2 = x_1 + h = .05 + .05 = .1$$

$$y_3 = 3.812 + (.05)\left[(.1)^2 + 4(.1) - \tfrac{1}{2}(3.$$

$$= \boxed{3.7377}$$

$$x_3 = x_2 + h = .1 + .05 = .15$$

$$y_4 = \boxed{3.6753}$$

$$x_4 = x_3 + h = .15 + .05 = .2$$

$$y_5 = \boxed{3.62}$$

N34

Example: Use Euler method for the I.V.P

$y' = xy$ with $y(0)=1$, $h = .2$ (compute y_1,

Solution:

$y' = f(x,y) = x.y$

$y_0 = 1, x_0 = 0, h = .2$

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_1 = y_0 + h f(x_0 y_0)$

$= 1 + .2 (0 \times 1) \Rightarrow y_1 = 1$

$x_1 = x_0 + h = 0 + .2 = .2$

$y_2 = y_1 + h (x_1 y_1) = 1 + .2(.2)(1) = 1.04$

$x_2 = x_1 + h = .2 + .2 = .4$

$y_3 = 1.04 + .2(.4 \times 1.04) = 1.123$

$x_3 = .4 + .2 = .6$

$y_4 = 1.1232 + .2(.6)(1.123)$

$= 1.2580$

Example: A ball at 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8),$$

$$\theta(0) = 1200 \text{ K}$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler method. Assume a step size of $h = 240$ seconds.

Solution: $$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

$$f(t, \theta) = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

by Euler method

$\theta_{i+1} = \theta_i + f(t_i, \theta_i) h$

For $i=0$, $t_0=0$, $\theta_0=1200$

$\theta_1 = \theta_1 + h\, f(t_0, \theta_0)$

$= 1200 + 240\, f(0, 1200)$

$= 1200 + 240(-2.2067 \times 10^{-12}(1200^4 - 81 \times 10^8))$

$= 1200 - (4.5579) \times 240 = 106.09\ K$

$t_1 = t_0 + h = 240 + 0 = 240$

$\theta_1 = \theta(240) \simeq 106.09\,K$

Let $i=1$, $t_1=240$, $\theta_1 = 106.09\,K$

$\theta_2 = \theta_1 + f(t_1, \theta_1) h$

$= 106.09 + f(240, 106.09) \times 240$

$= 106.09 + (-2.2067 \times 10^{-12}(106.09^4 - 81 \times 10^8)) \times$

$= 106.09 + (0.017595) \times 240 = 110.32\ K$

(37)

$t_2 = t_1 + h = 240 + 240 = 480$

$\theta_2 = \theta(480) \simeq 110.32\ K$

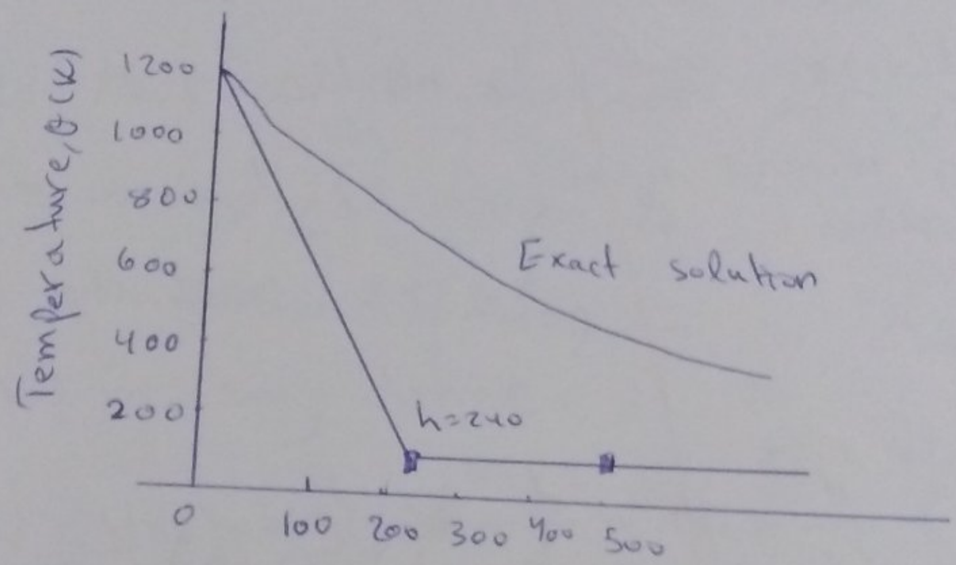


Fig. Comparing the exact solution and Euler method.

H.W: Use Euler method to solve the I.V.

$y' = 1+y$, $y(0)=1$, $h=0.1$ at $x=0.5$

Solution: $y_1 = 1.2$, $y_2 = 1.42$, $y_3 = 1.662$

$y_4 = 1.9282$

Interpolation

The Formula of Lagrange Interpolation is:

$x_1 \to y_1$
$x_2 \to y_2$
x_3

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\, y_1 + \cdots$$

$$+ \frac{(x-x_1)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_0-x_1)\cdots(x_n-x_{n-1})}\, y_n$$

multi   sum

$$\sum_{i=1}^{n} x_i$$

قانون

$$= \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{x-x_k}{x_j-x_k}\, y_k$$

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Example: Find the value of y at $x=0$ given some set of values $\overset{x_0\ y_0}{(-2,5)}$, $\overset{x_1\ y_1}{(1,7)}$, $(3,11)$, $(7,34)$?

① Lagrange
② n=data 4, x_0, y_0

Solution: The known values are:

$X=0$, $x_0=-2$, $x_2=3$, $x_3=7$, $y_0=5$, $y_1=7$, $y_2=11$, $y_3=34$

Using the interpolation formula:

$$y=\frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)}\times 5+\frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)}\times 7+$$

$$+\frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)}\times 11+\frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)}\times 34$$

$$y=\frac{21}{27}+\frac{49}{6}+\frac{-77}{20}+\frac{51}{54}=\frac{1087}{180}$$

Example

Find the value of y at $x=0$ given some set of values $(-2,6),(1,10),(3,12),(7,35)$?

Solution

∴ The known values are

$x=0$, $x_0=-2$, $x_1=1$, $x_2=3$, $x_3=7$, $y_0=6$

$y_1=10$, $y_2=12$, $y_3=35$

$$y=\frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)}\times 6+\frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)}\times 10$$

$$+\frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)}\times 12+\frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)}\times 35$$

$$y=\frac{14}{15}+\frac{35}{3}+\frac{-21}{5}+\frac{35}{36}$$

Example

Find the value of y at $x=0$ given some set of values $(-2,6),(1,10),(3,12),(7,35)$?

Solution: The known values are

$x=0$, $x_0=-2$, $x_1=1$, $x_2=3$, $x_3=7$, $y_0=6$

$y_1=10$, $y_2=12$, $y_3=35$

$$y=\frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)}\times 6+\frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)}\times 10$$

$$+\frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)}\times 12+\frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)}\times 35$$

$$y=\frac{14}{15}+\frac{35}{3}+\frac{-21}{5}+\frac{35}{36}$$

Least Square Method

The method of least square gives a way to find the best estimate, Using the following formula:

estimation

1) $\hat{Y} = a + bX$

n = no. of data

2) $b = \dfrac{n \sum XY - \sum X \sum Y}{n \sum X^2 - (\sum X)^2}$

$\sum x_i y_i = x_0 y_0 + x_1 y_1 + \dots$

3) $a = \bar{Y} - b\bar{X}$

$(\sum x_i)^2$

$\sum x_i = x_0 + x_1 + x_2 + \dots$

$\sum x_i^2 = x_0^2 + x_1^2 + \dots$

average

(52)

Example: Find an equation for the followin data using Least square method

X	1985	1986	1987	1988	1989	1990	1991	1992	19
Y	40	33	29	25	21	32	40	45	

1994
40

$n = 10$

L.S.M

Solution:

	X	Y	XY	X^2
1985	1	40	40	1
1986	2	33	66	4
1987	3	29	87	9
1988	4	25	100	16
1989	5	21	105	25
1990	6	32	192	36
1991	7	40	280	49
1992	8	45	360	64
1993	9	41	369	81
1994	10	40	400	100

$\Sigma X = 55$ $\quad \Sigma Y$ $\quad \Sigma XY$ $\quad \Sigma X^2 \Rightarrow$

$: 7 - \left(\frac{1}{2}\right)(2) - 0$

53

Now,

$$\bar{X} = \frac{55}{10} = 5.5$$

$$\bar{Y} = \frac{346}{10} = 34.6$$

$$\sum XY = 1999$$

$$\sum X^2 = 385$$

using $b = \dfrac{n\sum XY - \sum X \sum Y}{n\sum X^2 - (\sum X)^2}$

$$= \frac{10(1999) - 55 \times 346}{10(385) - (55)^2}$$

$$= \frac{19990 - 19030}{3850 - 3025} = \frac{960}{825}$$

$$= 1.16$$

and $a = \bar{Y} - b\bar{X}$

$$= 34.6 - 1.16(5.5)$$

$$= 28.22$$

$$\therefore \hat{Y} = 28.22 + 1.16X$$

Best estimation

$= 7 - (\frac{1}{2})(2) - 0$

Example: Find $\hat{Y}$ using LSM,

X	Y	X^2	XY
1	1	1	1
3	2	9	6
4	4	16	16
6	4	36	24
8	5	64	40
9	7	81	63
11	8	121	88
14	9	196	126
ΣX	$\Sigma Y =$	$\Sigma X^2 =$	$\Sigma XY =$

Now,

$$\Sigma X = 56, \quad \Sigma Y = 40, \quad \Sigma X^2 = 524, \quad \Sigma XY = 364$$

$$b = \frac{8(364) - (56)(40)}{8(524) - (56)^2} = \frac{7}{11} \simeq 0.636$$

$$\bar{X} = \frac{56}{8} = 7, \quad \bar{Y} = \frac{40}{8} = 5$$

$$a = \bar{Y} - b\bar{X} = 5 - \frac{7}{11}(7) = 0.545$$

$$\hat{Y} = 0.545 + 0.636X$$

55

H.W

Find an equation of least square line fitti[ng] the following data?

1945	98.2
1946	92.3
1947	80.0
1948	89.1
1949	83.5
1950	68.9
1951	69.2
1952	67.1
1953	58.3
1954	61.2

Interpolation

Newton forward difference :

Consider the function value (x_i, f_i), $i = 0, -, 5$, then the forward Difference Table is

Set of ordered paires

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
x_0	f_0				
		$\Delta f_0 = f_1 - f_0$			
x_1	f_1		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$		
		$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$	
x_2	f_2		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$		$\Delta^4 f_0 = \Delta^3 f_1 - \Delta^3 f_0$
		$\Delta f_2 = f_3 - f_2$		$\Delta^3 f_1 = \Delta^2 f_2 - \Delta^2 f_1$	
x_3	f_3		$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$		$\Delta^4 f_1 = \Delta^3 f_2 - \Delta^3 f_1$
		$\Delta f_3 = f_4 - f_3$		$\Delta^3 f_2 = \Delta^2 f_3 - \Delta^2 f_2$	
x_4	f_4		$\Delta^2 f_3 = \Delta f_4 - \Delta f_3$		
		$\Delta f_4 = f_5 - f_4$			
x_5	f_5				

ازواج

$$\Delta^5 f_0 = \Delta^4 f_1 - \Delta^4 f_0$$

forward قانون

Then, the n^{th} degree polynomial approximat for $f(x)$ is

$$f(x) \simeq f_0 + r\,\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \dots + \frac{r(r-1)\cdots(r-n)}{n!}\Delta^n f_0$$

$$+ \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0$$

Example: If $f(x)$ is known at the following data points:

x_i	0	1	2	3	4
f_i	1	7	23	55	109

to be found.

Then find $f(0.5)$ using Newton's forward difference formula:

Solution: Forward Difference Table:

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

equi-spaced

Then, By Newton's forward difference form

$$f(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$

at $x = 0.5$, $r = \frac{x - x_0}{h}$ $\Big\}$ Δ

$$= \frac{0.5 - 0}{1} = 0.5 \equiv r$$

$$f(0.5) = 1 + (0.5)(6) + \frac{(0.5)(0.5-1)\times 10}{2} + \frac{0.5(0.5-1)(0.5-2)\times 6}{6}$$

$$= 1 + 3 + 2.5(-0.5) + (-0.25)(-1.5) = 3.125$$

H.W Estimate $f(\overset{x}{1.5})$ for the data in the last example?

$r = \frac{x - x_0}{h}$??

- Newton – Gregory backward difference formula:

The following polynomial is called th[e] Newton – Gregory backward difference form[ula]

$$f(x) \simeq f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \cdots + \frac{s(s+1)(s+2)\cdots(s+n-1)}{n!}\nabla^n f_n$$

∇ backward

Δ forward

where $s = \dfrac{x - x_n}{h}$

equispaced

(61)

$$= 0.47943 + (-3.5)(0.09) + \frac{(-3.5)(-3.5+1)}{2!}(-0.0039)$$

$$+ \frac{(-3.5)(-3.5+1)(-3.5+2)}{3!}(-0.00035) +$$

$$\frac{(-3.5)(-3.5+1)(-3.5+2)+(-3.5+3)}{4!}(0.001217)$$

$$= 0.47943 - 0.315 - 0.01706 + 0.000765625$$

$$+ 0.0003386 = 0.14847$$

H.W

Example: Given the following data, estimate $f(4.12)$ using Newton-Gregory backward difference interpolation polynomial

i	0	1	2	3	4	5
x_i	0	1	2	3	4	5
f_i	1	2	4	8	16	32

$$S = \frac{X - X_n}{h} = \frac{4.12 - 5}{1}$$

$h = 4 - 3 = 1$

x_i	f_i	∇f_i	$\nabla^2 f$		
0	1				
		1			
1	2		1		
		2		1	
2	4		2		1
		4		2	
3	8		4		2
		8		4	
4	16		8		
		16			
5	32				

Solution: $x_n = 5$, $x = 4.12$, $h = 1$

$$\therefore S = \frac{x - x_n}{h} = \frac{4.12 - 5}{1} = -0.88$$

Newton Backward Difference polynomial $P_5(x)$ is given by:

$$P_5(x) \approx f_5 + S\nabla f_5 + \frac{S(S+1)}{2!}\nabla^2 f_5 + \frac{S(S+1)(S+2)}{3!}$$

$$+ \frac{S(S+1)(S+2)(S+3)}{4!}\nabla^4 f_5 + \frac{S(S+1)(S+2)(S+}{5!}$$

$S\nabla f$ $\quad \nabla f$ $\quad \frac{S(S-1)}{2!}\nabla^2 f$ $\quad \nabla^5 f_5$

$$= 32 + (-0.88)\,16 + \frac{(-0.88)(-0.88+1)}{2}\,8 +$$

S $\quad S-1$ $\quad S-2$ $\quad \nabla^3 f$

$$\frac{(-0.88)(-0.88+1)(-0.88+2)}{6}(4) +$$

$\nabla^3 f$

$$\frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)}{24 \; 3 \; 4!}(2) +$$

$$\frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3) + (-0.}{120}$$

(63)

$= 32 - 14.08 - 0.4224 - 0.07885 - 0.0209 - 0.$

$= 17.39135$

1. Formula & Examples

Formula

Newton's Forward Difference formula
$p = \dfrac{x - x_0}{h}$ unknown $y(x) = y_0 + p\Delta y_0 + \dfrac{p(p-1)}{2!}\cdot\Delta^2 y_0 + \dfrac{p(p-1)(p-2)}{3!}\cdot\Delta^3 y_0 + \dfrac{p(p-1)(p-2)(p-3)}{4!}\cdot\Delta^4 y_0 + \dfrac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y \dots$

Examples

1. Find Solution using Newton's Forward Difference formula

x	f(x)
1891	46
1901	66
1911	81
1921	93
1931	101

equi-distance

$X \equiv$ no.

$\Delta^4 y$

x = 1895

Solution:

The value of table for x and y

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Newton's forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	**46**				
		20			
1901	66		**-5**		
		15		**2**	
1911	81		-3		**-3**
		12		-1	
1921	93		-4		
		8			
1931	101				

Given

equi-

for

$h = 1931 - 1921$
$= 10$

The value of x at you want to find the $f(x)$: $x = 1895$

back

$h = x_1 - x_0 = 1901 - 1891 = 10$

$$p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Newton's forward difference interpolation formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \cdot \Delta^4 y_0$$

$$y(1895) = 46 + 0.4 \times 20 + \frac{0.4(0.4-1)}{2} \times -5 + \frac{0.4(0.4-1)(0.4-2)}{6} \times 2 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24} \times -3$$

$$y(1895) = 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$y(1895) = 54.8528$$

2. Find Solution using Newton's Forward Difference formula

x	f(x)
0	1
1	0
2	1
3	10

$y = \Delta^3 y$

\-

x = -1

Solution:
The value of table for x and y

x	0	1	2	3
y	1	0	1	10

Newton's forward difference interpolation method to find solution

Newton's forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	**1**			
		-1		
1	0		**2**	
		1		**6**
2	1		8	
		9		
3	10			

The value of x at you want to find the $f(x) : x = -1$

The value of x at you want to find the $f(x)$: $x = -1$

$h = x_1 - x_0 = 1 - 0 = 1$

$$p = \frac{x - x_0}{h} = \frac{-1 - 0}{1} = -1$$

$3! = 3 \times 2 \times 1$

Newton's forward difference interpolation formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0$$

$$y(-1) = 1 + (-1) \times -1 + \frac{-1(-1-1)}{2} \times 2 + \frac{-1(-1-1)(-1-2)}{6} \times 6$$

$$y(-1) = 1 + 1 + 2 - 6$$

$$y(-1) = -2$$

Solution of newton's forward interpolation method $y(-1) = -2$

Formula

Newton's Backward Difference formula
$p = \dfrac{x - x_n}{h}$ $y(x) = y_n + p \nabla y_n + \dfrac{p(p+1)}{2!} \cdot \nabla^2 y_n + \dfrac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n + \dfrac{p(p+1)(p+2)(p+3)}{4!} \cdot \nabla^4 y_n + \ldots$

Examples

1. Find Solution using Newton's Backward Difference formula

x	f(x)
1891	46
1901	66
1911	81
1921	93
1931	101

x = 1925

Solution:

The value of table for x and y

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

(Ent

Newton's backward difference interpolation method to find solution

Newton's backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		**-3**
		12		**-1**	
1921	93		-4		
		8			
1931	**101**				

y_n

h = any successive data

The value of x at you want to find the $f(x) : x = 1925$

$h = x_1 - x_0 = 1901 - 1891 = 10$

$$p = \frac{x - x_n}{h} = \frac{1925 - 1931}{10} = -0.6$$

on

p = forward = 0.4

p = backward = −0.6

Newton's backward difference interpolation formula is

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \cdot \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \cdot \nabla^4 y_n$$

$$y(1925) = 101 + (-0.6) \times 8 + \frac{-0.6(-0.6+1)}{2} \times -4 + \frac{-0.6(-0.6+1)(-0.6+2)}{6} \times -1 + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24} \times -3$$

$$y(1925) = 101 - 4.8 + 0.48 + 0.056 + 0.1008 = 96$$

4! = 4×3×2×1

2. Find Solution using Newton's Backward Difference formula

x	f(x)
0	1
1	0
2	1
3	10

x = 4

Solution:
The value of table for x and y

x	0	1	2	3
y	1	0	1	10

Newton's backward difference interpolation method to find solution

Newton's backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$4! = \nabla^3 y$

$x_n = 3$

$y_n = 10$

$h = 1$

The value of x at you want to find the $f(x) : x = 4$

$h = x_1 - x_0 = 1 - 0 = 1$

$$p = \frac{x - x_n}{h} = \frac{4 - 3}{1} = 1$$

Newton's backward difference interpolation formula is

$$y(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \cdot \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n$$

$$y(4) = 10 + 1 \times 9 + \frac{1(1+1)}{2} \times 8 + \frac{1(1+1)(1+2)}{6} \times 6$$

$$y(4) = 10 + 9 + 8 + 6$$

$$y(4) = 33$$

Solution of newton's backward interpolation method $y(4) = 33$

Tylor's Method series method :

Consider the one-dimensional I.V.P

$$y' = f(x,y) \,,\; y(x_0) = y_0 \quad , \text{wher}$$ } given

f is a function of two variables x and y, and (x_0, y_0) is a Known point the solution curve.

Then we may define : قانون مهم :

المطلوب

$$y(x_0+h) = y(x_0) + h\, y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

معطى

$2! = 2\times 1$ $\quad 3! = 3\times 2\times 1$

where $y' = f(x,y)$

$$y'' = f_x + f_y\, y'$$ $\quad , y''' = f_{xx} + f_{xy} y' + f_y y'' + y'$

$$y''' = f_{xx} + 2 f_{xy}\, y' + f_{yy} (y')^2 + f_y\, y''$$

and so on , then we may write

$$y(x_0+h) = y(x_0) + h f + \frac{h^2}{2!} (f_x + f_y\, y') + \frac{h^3}{3!} \big(f_{xx} + 2 f_{xy} + f_{yy}\, y'^2 + f_y\, y'' \big) + O(h^4)$$

order 4

Example: Solve the intial value problem

$f(x,y) \equiv y' = -2xy^2, \; y(0)=1$ for y at $x=1$

with step length h = 0.2 using Taylor

method of order 4.

Solution: Given $y' = f(x,y) = -2xy^2 \Longrightarrow$

$y'' = -2[x\,2y \cdot y' + y^2]$

$y''' = -4[xyy'' + y'(xy'+y)]$

$$y'' = -2y^2 - 4xyy'$$

$y'' = -2y^2 - 4xy(-2xy^2)$

$= -2y^2 + 8x^2y^3$

$$y''' = -8yy' - 4xy'^2 - 4xyy''$$

$\cancel{(y')^2}$ y''

$$y^{iv} = -12y'^2 - 12yy'' - 12xy'y'' - 4xyy'''$$

$$y^{v} = -48y'y'' - 16yy''' - 12x(y'')^2 - 16xy'y''' - 4xyy^{iv}$$

The forth order Taylor's formula is

$$y(x_i+h) = y(x_i) + hy'(x_i,y_i) + \frac{h^2\, y''(x_i,y_i)}{2} + \frac{h^3}{3!}y'''(x_i,y_i) + \frac{h^4}{4!}y^{iv}(x_i,y_i) + \dots$$

given $x_0 = 0, y = 1, h = .2 \Rightarrow$

$y' = -2(0)(1)^2 = 0 \Rightarrow f = -2xy^2$

$y'' = -2(1)^2 - 4(0)(1)(0) = -2$

$y''' = -8(1)(0) - 4(0)(0)^2 - 4(0)(1)(-2) = 0$

$y^{iv} = -12(0)^2 - 12(1)(-2) - 12(0)(0)(-2) - 4(0)(1)(0)$

$= 24$

$\cancel{y(0.2) = 1 + 0.2(0) - 12(1)(-2) - 12(0)(0)(-2)}$

$\cancel{-4(0)(1)(0)}$

$y(\overbrace{0.2}^{x_0+h}) = 1 + .2(0) + (.2)^2(-2)/2! + 0 + .2^4(24)/4!$

$= \underline{\underline{0.9615}}$

now at $x = 0.2$, we have $y = 0.9615$

$y' = -0.3699$, $y'' = -1.5648$, $y''' = 3.9397$

and $y^{iv} = 11.9953$

$\Rightarrow y(0.4) = 1 + .2(-.3699) + (.2)^2(-1.5648)/2!$

$+ (.2)^3(3.9397)/3! + (0.2)^4(11.9953)/4! = \underline{\underline{0.862}}$

$y(0.6) = 1 + 0.2(-0.5950) + 0.2^2(-0.6665)/2!$

$+ 0.2^3(4.4579)/3! + 0.2^4(-5.4051)/4!$

$= 0.7356$

$y(0.8) = 0.6100$

$y(1) = 0.5001$

$\therefore$ at $x = 1$ we have $y = 0.5001$

Example: Use Taylor series method to find $y(\underbrace{0.1}_{h})$ for $y' = x - y^2$, $y(0) = 1$ correct up to 4 decimal places?

$\boxed{y'^2 \; ? \; y''}$

Solution: Given $y' = x - y^2$

$\Rightarrow \quad y'' = 1 - 2yy' \Longrightarrow = 1 - 2y(x - y^2)$

$= 1 - 2xy + 2y^3$

$\Rightarrow y''' = -2yy'' - 2y'^2$

$y^{iv} = -2yy''' - 6y'y''$, $y^{v} = -186$

at $x = 0$, $y = 1 \Rightarrow$

$y''' = -2(xy' + y) + 6y^2 \cdot y'$

$y' = -1, \quad y'' = 3, \quad y''' = -8, \quad y^{iv} = 34$

The 4^{th} order Taylor's formula is

$$y(x) = y(x_0) + \overbrace{(x - x_0)}^{h} y'(x_0, y_0) + (x - x_0)^2 y''(x_0, y_0)/2!$$

$$+ \frac{(x-x_0)^3}{3!} y'''(x_0, y_0) + \frac{(x-x_0)^4}{4!} y^{iv}(x_0, y_0) + \cdots$$

Now

$$y(.1) = 1 - (0.1) + 3(0.1)^2/2 - 4(0.1)^3/3 + 17(0.1)^4/12$$
$$- 31(0.1)^5/20$$

$$= 0.9 + 3(0.1)^2/2 - 4(0.1)^3/3 + 17(0.1)^4/12 - 31(0.1)^5/2$$

$$= 0.9137 + 17(0.1)^4/12 - 31(0.1)^5/20$$

$$= 0.9138 - 31(0.1)^5/20$$

$$= \underline{\underline{0.9138}}$$

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H.W

Using Taylor series method of order 4 to solve the I.V.P. $y' = \frac{x-y}{2}$ on $[0, 3]$ with $y(0) = 1$. Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$

$y' = f(x,y) = \frac{1}{2}(x - y)$

$y'' = \frac{1}{2}(1 - y')$

$y'' = \frac{1}{2}\left(1 - \frac{x-y}{2}\right)$

$= \frac{1}{2} - \frac{1}{4}(x-y)$

derivatives

Note The final answers are:

x_i	h=1	h=0.5	h=0.25
0	1	1	1
0.125			
0.250			0.897492
0.375			
0.5		0.836426	0.836404
0.750			0.811870
1	0.820315	0.819629	0.819592
1.5		0.917142	0.917100
2	1.104513	1.103683	1.103641
2.5		1.359558	1.359517
3	1.670186	1.669431	1.669393

(46)

(2) Simpson's Rule:

$y_1, y_1 \to y_1$, $i=2$, y_2, y_3, y_4, y_5

Here our formula would be

$$\text{Area} = \int_a^b f(x)\,dx = \frac{h}{3}\left[y_1 + 4\sum_{i=2,4,6}^{i=n} y_i + 2\sum_{i=3,5,7}^{n-1} y_i + y_{n+1}\right]$$

قانون

Example: Evaluate $\int_0^{\pi/2} \sin x \, dx$ using Simpson rule with $n=4$?

$h = \frac{\frac{\pi}{2} - 0}{4}$

Solution: $h = \frac{b-a}{n} \Rightarrow h = \frac{\pi}{8}$

x	$f(x) = y_i$	
0	$\sin(0) = 0$	y_1
$\frac{\pi}{8}$	$\sin(\frac{\pi}{8}) = 0.38268$	y_2
$\frac{\pi}{4}$	$\sin(\frac{\pi}{4}) = 0.70711$	y_3
$\frac{3\pi}{8}$	$\sin(\frac{3\pi}{8}) = 0.92388$	y_4
$\frac{\pi}{2}$	$\sin(\frac{\pi}{2}) = 1.0$	y_5

$y_1 \to a$ is

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$$\text{Area} = \frac{\pi/18}{3}\left[y_1 + 4(y_2 + y_4) + 2(y_3) + y_5\right]$$

$$= \frac{\pi}{24}\left[0 + 4(0.38268 + 0.92388) + 2(0.70711) + 1\right]$$

$$= 1.001$$

H-W

Evaluate The following integrals using Simp
rule

1- $\int_1^3 e^{\cos(x)}\, dx$, with $n = 4$

Answer 1.5826

$\Rightarrow h?$

$x \quad y$

y_1

y_2

y_3

y_4

y_5

ver

Area $\frac{h}{3}[\quad]$

Example: Use Simpson's rule with $n=6$ to estimate $\int_{a\,1}^{b\,4} \sqrt{1+x^3}\,dx$. Compute $(y_0, \ldots, y_6)$

Solution: $n=6 \implies h = \frac{4-1}{6} = \frac{1}{2}$ ✓

x	y	
1	$\sqrt{2}$	$\implies \sqrt{1+1^3} = \sqrt{2}$ y_0
1.5	$\sqrt{4.375}$	$\implies \sqrt{1+(1.5)^3}$ (2)
2	3	y_3
2.5	$\sqrt{16.625}$	(4)
3	$\sqrt{28}$	y_5
3.5	$\sqrt{43.875}$	(6)
4	$\sqrt{65}$	

Therefore $\cancel{\int_1^4 \sqrt{1+x^3}\,dx \approx \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2\cdot 3 + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43}\ldots\right)}$

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Therefore

$$\int_1^4 \sqrt{1+x^3} \simeq \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2(3) + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43.875} + \sqrt{65}\right)$$

$$\simeq 12.871$$