Matrices matricies المصفوفات

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called entries (or sometimes elements) of the matrix.

For example

$$\begin{array}{cc} & \text{column 1} \\ \text{Row 1} \rightarrow & \\ \text{Row 2} \rightarrow & \end{array} \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}_{2\times 3} , \quad \begin{array}{c} C_1 \\ \begin{bmatrix} 4 \\ 1/2 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \end{array} \; 2\times 1 , \quad \begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix} ,$$

are all matrices.

- The first matrix has two rows and three columns (we write 2×3)
- The second matrix has just a single column; it is called vector (2×1)
- The third matrix is a square matrix

1

i.e. it has as many rows as columns
(we write 2×2)

Example: In a system of linear equations, briefly called a linear system, such as

$$\left.\begin{array}{l} 4x_1 + 6x_2 + 9x_3 = 6 \\ 6x_1 \qquad - 2x_3 = 20 \\ 5x_1 - 8x_2 + x_3 = 10 \end{array}\right\}$$

$A_{2 \times 2}$

$A_{3 \times 3}$

The coefficients of the unknowns x_1, x_2, x_3 are the entries of the coefficient matrix, call it A,

$$A = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & +1 \end{bmatrix}$$

But, when we need to solve the system, we must write down the augmenting matrix, where we augmenting A by the right sides of the linear system, as follows:

③

R.H.S

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array}\right]$$

a_{32}, a_{23}, a_{22}

-8 $\quad -2 \quad 0$

Def: Two matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ are equal, written $A = B$, if and only if they have the same size and the corresponding entries are equal, that is

$a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.

Example: Let $\underset{2\times 2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and

$\underset{2\times 2}{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$; Then

$A = B$ iff

$a_{11} = 4 \qquad a_{12} = 0$

$a_{21} = 3 \qquad a_{22} = -1$

Def

— : The sum of two matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ of the same size is written $A+B$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of A and B.

of the same sizes

Matrices of different sizes cannot be added.

Example

— : If $A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ 2×3, $B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$ 2×3,

then $A+B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ " " 2×3

$a_{11}+b_{11}$ $a_{12}+b_{12}$ ...

$a_{21}+b_{21}$

Def

— : The product of any $m\times n$ matrix $A = [a_{jk}]$ and any scalar c (number) is written cA and is the $m\times n$ matrix

constant

$cA = [c a_{jk}]$ obtained by multiplying each entry of A by c.

Example: If $A = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$

Then $-A = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$

Rules for matrix Addition and Scalar Multiplication

constant or

$$A + B = B + A$$
$$(A+B)+C = A+(B+C)$$
$$A + 0 = A \quad ; 0 \text{ is zero matrix}$$
$$A + (-A) = \boxed{0}$$
$$c(A+B) = cA + cB$$
$$(c+k)A = cA + kA$$
$$c(kA) = (ck)A$$
$$IA = A$$

$A_{2\times3}$, $A_{4\times1}$ ✗

I : identity matrix → square; → 1, 0

$, I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$A_{2\times2}$, $A_{3\times3}$, $A_{10\times10}$

diagonal

off diagonal

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

H.W

Let $A = \begin{bmatrix} 3 & 0 & 4 \\ -1 & 2 & 2 \\ 6 & 5 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -5 & -3 \\ -5 & 2 & 4 \\ -3 & 4 & 0 \end{bmatrix}$

$u = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ $v = \begin{bmatrix} -4.5 \\ .8 \\ 1.2 \end{bmatrix}$

system $\rightarrow$ matrix

system $\leftarrow$ matrix

1- Find the following expressions

$2(A+B)$, $2A+2B$, $33u$, $4v+9u$, $u-v$

2- Write down a linear system whose augmented matrix is the matrix B.

$A_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = Identity matrix

a_{11}, a_{22}, a_{33}

Dimension

size A, B
3x1 2x2

Matrix Multiplication:

Def: The product $C = AB$ (in this order) of an $m \times n$ matrix $A = [a_{jk}]$ times an $r \times p$ matrix $B = [b_{jk}]$ is defined iff

$r = n$ and is then $m \times p$ matrix $C = [c_{jk}]$ with entries:

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk}$$

A 3x2   B = 2x1

$$= a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk}$$

$A_{m \times k} \cdot B_{k \times n}$

$j = 1, \ldots, m$
$k = 1, \ldots, p$

Note: The condition $r = n$ means that the 2nd factor, B, must have as many rows as the first factor has columns, as shown

$$\begin{array}{ccc} A & B & = C \\ [m \times n] & [n \times r] & [m \times r] \end{array}$$

c_{jk} in (Def above) is obtained by multiplying each entry in the j^{th} row of A by the corresponding entry in the k^{th} column of B and then adding these n products.

Example: $AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$

$3\times3 \qquad 3\times4$

c_{11}: $3\times2 + 5\times5 + (-1)9$

$6 + 25 - 9$

c_{12}: $3\times(-2) + 5(0) + (-1)(-4)$

$= -6 + 4 = -2$

$= \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$

$c_{13} = 3\times3 + 5\times7 + (-1)(1)$

$= 9 + 35 - 1$

$= 43$

?

Example: $\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$

$2\times2 \quad 2\times1 \implies C_{2\times1}$

Note $\rightarrow$ $AB \neq BA$ in general

$C = \begin{bmatrix} 4\times3 + 2\times5 \\ 1\times3 + 8\times5 \end{bmatrix} = \begin{bmatrix} 12 + 10 \\ 3 + 40 \end{bmatrix} = \begin{array}{c} C_1 \\ \begin{bmatrix} 22 \\ 43 \end{bmatrix} \\ 2\times1 \end{array} \begin{array}{l} R_1 \\ R_2 \end{array}$

$A_{3\times2} \cdot B_{2\times1}$ / $B_{2\times1} \cdot A_{3\times2}$ X

$C_{3\times1}$

Rules of Matrix Multiplications

$(kA)B = k(AB) = A(kB)$, k is a scalar

$A(BC) = (AB)C$

$(A+B)C = AC + BC$

$C(A+B) = CA + CB$

Def: The Transpose of an $m \times n$ matrix $A = [a_{jk}]$ is the $n \times m$ matrix, A^T (read A transpose) that has the first row of A as its first column, the 2^{nd} row of A as its 2^{nd} colum, and so on. Thus the transpose of A is $A^T = [a_{kj}]$; i.e

$$A^T = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & & & \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$A_{3\times 2} \Rightarrow A_{2\times 3}$

Example

- If $A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}_{2\times3}$, then $\Longrightarrow$

$$A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}_{3\times2}$$

- Also, $\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^T_{3\times1} = \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}_{1\times3} \rightarrow$ 3×1

Rules for transposition are

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

c 1×1

$[6]$?

d_{11}

d_{22}

$A_{2\times3}$, B $= {}_{2\times4}]^T$

$C_{4\times2}$

$A^T_{3\times2}$ $B^T_{4\times3}$

$B^T_{4\times3}$, $A^T_{3\times2}$

Special Matrices

Transposition gives rise to two usefu[l]
classes of matrices, as follows.

① Symmetric and skew-symmetric matri[ces]
are squar matrices whose transpose equ[als]
the matrix it self or minus the mat[rix]
respectively:

$$A^T = A, \quad (\text{thus } a_{kj} = a_{jk})$$

$$A^T = -A \quad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0)$$

Example

$$A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \text{ is symmetri[c]}$$

and

$$B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \text{ is skew-symm[etric]}$$

Lower Upper

1) Transpose

1) Sizes

2) Summation and subtraction

3) multiplication

(2) Triangular Matrices

- Upper triangular matrices are square matrices that can have non_zero entries only on and above the main diagonal, whereas any entry below the diagonal must be zero.

Similarly,

- Lower triangular matrices can have nonzero entries only ~~and~~ on and below the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

Example:

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} ; \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

2×2 3×3

Upper triangular

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix} \quad 3 \times 3 \qquad , \qquad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix} \quad 4 \times 4$$

Lower triangular

(3) Diagonal Matrices

These are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

If all the diagonal entries of a diagonal matrix S are equal, say c, we call S a scalar matrix because multiplication of any square matrix A of the same size S has the same effect as the multiplication by a scalar, that is

$$AS = SA = cA$$

In particular, a scalar matrix whose entries on the main diagonal ~~of~~ are all

1 is called a unit matrix (or identity matrix) and is denoted by I_n or simply I, and we have

$$IA = AI = A$$

non-zero $AB \neq BA$ in general

Example

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \text{Diagonal matrix}$$

$$S = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \equiv \text{Scalar matrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \text{Identity (Unit) matrix}$$

H.W

Let $A = \begin{bmatrix} 6 & -2 & -2 \\ 10 & -3 & 1 \\ -10 & 5 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 9 & 4 & -4 \\ 4 & 7 & 0 \\ -4 & 0 & 11 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 1 \\ 0 & -2 \\ 4 & 0 \end{bmatrix}$, $a = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$, $b = [3 \quad 0 \quad 8]$

A^2 ? $A \cdot A = C$

Find the following

1- AB, BA, AA^T, A^TA

2- A^Tb, b^TB, $(3A-2B)^Ta$

C^3

3- A^2B, A^3, $(AB)^2$, A^2B^2

4- a^TCC^Ta, bCC^Tb^T

(1) det(A)
(2) D
(3) |A|

Determinants

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then 2×2

ضرب

$\det(A) = a_{11}a_{22} - a_{12}a_{21}$ قانون

Example

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, then

$3 - 8$

$\det A = (1)(3) - (2)(4) = -5$ free

- Let $A = \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ 3×3

(1) a_{12}

To find the det, we do the following

قانون 3×3

det, row 1

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$- a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(2)

Example: Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & \boxed{1} & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}_{3\times 3}$$

Solution:

$$\det A = 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(\boxed{-1+6}) - 1(3+4) + 3(9+2)$$

$$= 36$$

$$|A| = 2\left[(-1)(1) - (-2)3\right] - \left[3\cdot 1 - (-2)(2)\right] + 3\left[3\cdot 3 - (-1)(2)\right]$$

$$= 2\left[-1-(-6)\right] - \left[3-(-4)\right] + 3\overset{11}{\boxed{9-(-2)}}$$

$$= 2[5] - [7] + 33 = 10 - 7 + 33$$

$$= 3 + 33 = 36$$

Cramer's Rule

It is a rule for solving a system of linear equations like:

$$\overbrace{a_{11}x + a_{12}y}^{} = \overbrace{b_1}^{R.H.S} \quad \text{--- (1)}$$

$$a_{21}x + a_{22}y = b_2 \quad \text{--- (2)}$$

1

If $D = \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

then the solution of the equations can be obtained as follows

2

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

Example : Solve the system of eqs using Cramer's rule

$$3x_1 - 2x_2 = 6$$
$$2x_1 + x_2 = 0.5$$

Solution :

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-2)(2) = 3 - (-4) = 7$$

$$= 3 + 4 = 7$$

$$x_1 = \frac{1}{7} \begin{vmatrix} 6 & -2 \\ .5 & 1 \end{vmatrix} = \frac{1}{7}[6 + 1] = 1$$

$$x_2 = \frac{1}{7} \begin{vmatrix} 3 & 6 \\ 2 & 0.5 \end{vmatrix} = \frac{1}{7}[1.5 - 12] = -1.5$$

. As for a system of three eqs with three unknown like:

R.H.S

$$
\left.
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\
a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3
\end{aligned}
\right\}
$$

Crammer's

The solution is

$$x_1 = \frac{1}{D} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{1}{D} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{1}{D} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example: Use Cramer's rule to solve the following linear system:

$$2x + y - z = 2$$
$$x - y + z = 7$$
$$2x + 2y + z = 4$$

arranged ?

Solution:

1

$$D = \begin{vmatrix} 2 & \boxed{1} & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{vmatrix}$$

a_{12} $-$

$- \rightarrow +$

$+ \rightarrow -$

$2((-1)(1) - (1)(2))$

$-1 - 2$

$$= 2\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= 2(-1-2) - 1(1-2) - 1(2+2)$$

$$= -6 + 1 - 4 = -9 \quad D$$

$$x_1 = \frac{1}{-9}\begin{vmatrix} 2 & \boxed{1} & -1 \\ 7 & -1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \frac{-1}{9}\left[2\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 7 & 1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 7 & -1 \\ 4 & 2 \end{vmatrix}\right]$$

$$= \frac{-1}{9}\left[2(-3) - 1(3) - 1(18)\right]$$

$$= \frac{-1}{9}(-6-3-18) = \frac{-27}{-9} = 3$$

first unknown

$$x_2 = \frac{-1}{9}\begin{vmatrix} 2 & \boxed{2} & -1 \\ 1 & 7 & 1 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= \frac{-1}{9}\left[2\begin{vmatrix} 7 & 1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 7 \\ 2 & 4 \end{vmatrix}\right]$$

$$= 2(3) - 2(-1) - (-10)$$

$$= 6 + 2 + 10 = 18$$

(22) $X_3 = \frac{-1}{9}\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ 2 & 2 & 4 \end{bmatrix}$, $+2 \mid 1$

$X_2 = -2$, $X = 2$ $= \frac{-1}{9}[21 \ 1 - 1]$ 4 Det

Minors and cofactors

$A_{3\times 3}$

If we have $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ minor / cof

Then, $A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

A_{11} = The minor of a_{11} is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of a_{12} is $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

⋮

A_{33}

The minor of a_{33} is $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

A_{11}, A_{12}, A_{13}
A_{21}, A_{22}, A_{23}
A_{31}, A_{32}, A_{33}

The cofactor of a_{ij} is the det A_{ij}

that is $(-1)^{i+j}$ times the minor of a_{ij},

Thus,

$A_{11} \leftarrow$

i = rows
j = colomns

$(i+j) \to$ odd $\to -$
$(i+j) \to$ even $\to +$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

cofactor

$\therefore$ det $A = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$ inverse

Example

Find det (A) where $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$ 3×3

Solution

$(-1)(1) - (-2)(3)$ $= -1 + 6$

cofactors $A_{11} = (-1)^{2} \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} = -1 + 6 = 5$

$3 - (-4) = 7$ $+5$ minor

$A_{12} = (-1)^{3} \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = -(3+4) = -7$ cofactor

$i+j$

$A_{13} = (-1)^{4} \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = 9 + 2 = 11$ $(-1)^{2} = +1$

inverse $i+j$ = even 3

$(-1)^{1+2} = (-1)$

det $A = (2)(5) + (1)(-7) + 3(11)$ $(-1)(-1)(-1)$

$= 10 - 7 + 33 = 36$

(25) $A_{32} = \begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = 6-(-4) = 10 \to -10$

$\underbrace{32}_{5 \to \text{odd}}$

Example: Let $A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$

find A^{-1} $A_{3\times3}$

$A^{-1} \to |A|$, $\to$ cofactors, $\to A^{-1}$

Solution

det $A = a_{11} \underbrace{A_{11}}_{\text{cofactor}} + a_{12} A_{12} + a_{13} A_{13}$ } → det inverse

$-2-(-3)$

even

$\underbrace{A_{11}}_{} = (-1)^2 \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = -2+3 = 1$

-10

$A_{\underbrace{12}_{\text{odd}}} = (-1)^3 \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = -(-1-9) = 10$

$A_{13} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1-6 = -7$

$A_{13} = -7$ ~~A_{33}~~

- det $A = 2(1) + (3)(10) + (-4)(-7)$

$= 60 \neq 0$ (a) / 3 cofactor

compute $A_{21}, A_{22}, \cdots$ 6 are left ~~to be found~~

- $\text{cof } A = \begin{bmatrix} 1 & 10 & -7 \\ 7 & 10 & 11 \\ 17 & -10 & 1 \end{bmatrix}$ ← A_{21} A_{22} A_{23} ← A_{31} A_{32} A_{33}

2nd Row

$A_{21} = (-1)^3 \begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} = 7$ $\{3)(-1) - (-4)(-1)\}$

$-3 - 4 = -7$

موجب
وزي

$A_{22} = (-1)^4 \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} = 10$

$A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 11$

Third Row

$A_{31} = (-1)^4 \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 17$

$A_{32} = (-1)^5 \begin{vmatrix} 2 & -4 \\ 1 & 3 \end{vmatrix} = -10$

$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$

(24)

1) Minors → det
2) cofactors matrix
~~3) Adj [cofac]^T~~
4) $A^{-1} = \frac{1}{\det}$ adj

Inverse of a Matrix

To find the inverse of a square matrix A, we must check first that $\det(A) \neq 0$, then

- Construct the matrix of cofactors of A.

$$\text{cof } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- Construct the transpose matrix of cofactors (which is called the adjoint of A)

i.e

$$\text{adj } A = (\text{cof } A)^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

adjoint

- Then $A^{-1} = \frac{1}{\det A} \text{adj } A$ قانون

A^{-1} is called the inverse of A

$$\text{adj } A = \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix} = \text{cof}(A)^T$$

$$A^{-1} = \frac{1}{60} \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

H.W

- Use Cramer's rule to find the solution for

2×2

① $\rightarrow 4x - 3y = 6$

$\rightarrow 3x - 2y = 5$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{bmatrix}$$

3×3

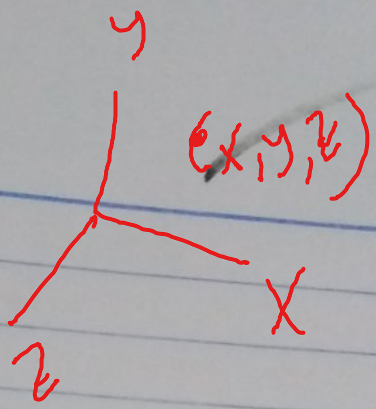
② $\begin{cases} x - z = 3 \\ 2y - 2z = 2 \\ 2x + z = 3 \end{cases}$

x y z

$$A^{-1} = \frac{1}{5} \begin{bmatrix} -6 & 18/5 & 7/5 \\ 5 & -3 & -1 \\ -1 & 4/5 & 1/5 \end{bmatrix}$$

- Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$

Find A^{-1}



Vectors

Def: The general form of a vector in cartisian coordinates (x,y,z) is:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \text{in 3-D}$$

(3,0,-2)

where

constants

$\overbrace{A_x}^{3}$, A_y, A_z are components of

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors

Examples: ① $\vec{A} = 2\hat{i} + \hat{j} + 5\hat{k}$ componts

$A_x = 2$, $A_y = 1$, $A_z = 5$

② $\vec{A} = 3\hat{i} - \hat{k} + 0\hat{j}$

$A_x = 3$, $A_y = 0$, $A_z = -1$

Addition and Subtraction of Vectors

جمع المتشابه

Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

vector

$\therefore \vec{A} \pm \vec{B} = (A_x \pm B_x)\hat{i} + (A_y \pm B_y)\hat{j} + (A_z \pm B_z)\hat{k}$

نجمع او (نطرح) المتشابهات

Notes

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
2. $\vec{A} - \vec{A} = \vec{0}$
3. $\vec{A} - \vec{B} = -(\vec{B} - \vec{A})$

Examples

$\vec{A} = 2\hat{i} - 3\hat{j} + 5\hat{k}$

$\vec{B} = \hat{i} + 6\hat{j} - 2\hat{k}$

Find $\vec{A} + \vec{B}$, $\vec{A} - \vec{B}$

$\vec{A} - \vec{B} = \hat{i} - 9\hat{j} + 7\hat{k}$

$\Rightarrow (2+1)\hat{i} + (-3+6)\hat{j} + (5+(-2))\hat{k}$

$(\vec{A} - \vec{B}) = 2\hat{i} - 3\hat{j} + 5\hat{k} - (\hat{i} + 6\hat{j} - 2\hat{k})$

Solution: $\vec{A}+\vec{B} = 3\hat{i}+3\hat{j}+3\hat{k}$

$\vec{A}-\vec{B} = \hat{i}-9\hat{j}+7\hat{k}$

Example: $\vec{A} = \hat{i}+2\hat{j}-5\hat{k}$

$\vec{B} = 10\hat{i}+\hat{k}$ $+0\hat{j}$

Find $\vec{B}-\vec{A}$, $\vec{A}-\vec{B}$

$\vec{B}-\vec{A} = 10\hat{i}+\hat{k} - (\hat{i}+2\hat{j}-5\hat{k})$

Solution: $\vec{B}-\vec{A} = (10-1)\hat{i}+(0-2)\hat{j}+(1+5)\hat{k}$

$= 9\hat{i}-2\hat{j}+6\hat{k}$

$\vec{A}-\vec{B} = -(9\hat{i}-2\hat{j}+6\hat{k})$

$= -9\hat{i}+2\hat{j}-6\hat{k}$

Note: $n\vec{A} = nA_x\hat{i}+nA_y\hat{j}+nA_z\hat{k}$

where n is constant: $+ve$, $-ve$

Length of vector

① If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\sqrt{(-2)^2}$

Then

$A_x = -2$

$$|\vec{A}| = \|\vec{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$

$\hat{i}\,\hat{j}\,\hat{k}$

is length (or magintude) of $\vec{A}$.

constant $\neq$ −ve

② The unit vector of $\vec{A}$ is

vector $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$ $\quad \hat{i}\,\hat{j}\,\hat{k}$ $\quad \hat{i}$

Example, $\vec{A} = -2\hat{i} + 5\hat{j} - \hat{k}$ } Find $\hat{A}$?

$|\vec{A}| = (-2)^2 + (5)^2 + (-1)^2$

$= \sqrt{4 + 25 + 1} = \sqrt{30}$

$$\hat{A} = \frac{-2\hat{i} + 5\hat{j} - \hat{k}}{\sqrt{30}} = \frac{-2}{\sqrt{30}}\hat{i} + \frac{5}{\sqrt{30}}\hat{j} - \frac{1}{\sqrt{30}}\hat{k}$$

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$\vec{A} \cdot \vec{B} = (A_x \cdot B_x) + (A_y \cdot B_y) + (A_z \cdot B_z)$ const ant

$\vec{A} \cdot \vec{B}$ نقطي (dot) or scalar

Example

$\vec{A} = 1\hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{B} = -3\hat{i} + 6\hat{j} + 2\hat{k}$

Find $\underbrace{\vec{A} \cdot \vec{B}}_{(1)}$, $\underbrace{\text{The angle between } \vec{A} \, \& \, \vec{B}}_{(2)}$

Solution

$\vec{A} \cdot \vec{B} = \overset{-3}{(1)(-3)} + \overset{-12}{(-2)(6)} + \overset{+4}{(2)(2)}$

$= -11$ scalar

$\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \dfrac{-11}{\sqrt{1+4+4}\,\sqrt{9+36+4}}$

$= \dfrac{-11}{21}$

كيف نجيب θ?

$3 \quad \sqrt{49}$

$\cos\theta = n$

$\cos^{-1}(\cos\theta) = \cos^{-1}(n)$

$\therefore \theta = \cos^{-1}\left(\dfrac{-11}{21}\right)$

$\simeq 121.6°$ ✓

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Product → constant
→ ~~vector~~

ضرب، نقطي

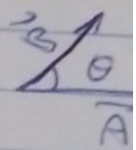
القياسي العادي

The Scalar (or Dot) Product

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$, so the result

$\vec{A} \cdot \vec{B}$ is scalar

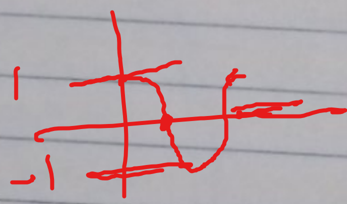
$\vec{A} \cdot \vec{B} = (A_x \cdot B_x) + (A_y \cdot B_y) + (A_z B_z)$



Notes

① $\vec{A} \cdot \vec{A} = A^2$

Cos(y)



② $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

③ $\vec{A} \cdot \vec{B} = 0$ at $\theta = 90^\circ, 270^\circ$

④ $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$ at $\theta = 0, 360^\circ, \ldots$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

Example

$\vec{A} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{B} = -3\hat{i} + 6\hat{j} + 2\hat{k}$

Find $\vec{A}\cdot\vec{B}$, The angle between $\vec{A}$ & $\vec{B}$

Solution

$$\vec{A}\cdot\vec{B} = (1)(-3) + (-2)(6) + (2)(2)$$

$$= -11$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-11}{\sqrt{1+4+4}\,\sqrt{9+36+4}}$$

$$= \frac{-11}{21}$$

1.

(ans

$\cos^{-1}(\cos(\theta)) =$

$$\therefore \theta = \cos^{-1}\left(\frac{-11}{21}\right)$$

$$\simeq 121.6^\circ$$

② $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

outkom

The Vector (or $\overbrace{\text{Cross}}^{\times}$) Product

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \equiv \vec{C}$$

the result is vector

Notes:

① $\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}|$ if $\theta = 90^\circ$

$\vec{A} \times \vec{B} = 0$ if $\theta = 0, 180^\circ, 360^\circ$

$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

(2) (3) (1)

$A \times (\vec{B} \times \vec{C}) = \underbrace{(\vec{A} \cdot \vec{C})}_{\text{constant}}\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Example : Let $\vec{A} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{B} = 3\hat{i} + \hat{k} = 3\hat{i} + 0\hat{j} + \hat{k}$

Find $\vec{A} \times \vec{B}$, $\vec{B} \times \vec{A}$

Solution

a_{12}

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$i[2-0]$ $[1+6]j$

$$= \hat{i} \left[(2)(1) - (-2)(0)\right] - \hat{j}\left[(1)(1) - (-2)(3)\right]$$

$[0-6]\hat{k}$

$$+ \hat{k}\left[(1)(0) - (2)(3)\right]$$

$$= 2\hat{i} - 7\hat{j} - 6\hat{k}$$

vector

$-(A \times B)$

$$\vec{B} \times \vec{A} = -(2\hat{i} - 7\hat{j} - 6\hat{k})$$

$$= -2\hat{i} + 7\hat{j} + 6\hat{k}$$

$$\vec{B} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 1 \\ 1 & 2 & -2 \end{vmatrix}$$

$= i(-2) - j(-7)$

$+ \hat{k}(6)$

$= -2\hat{i} + 7\hat{j} + 6\hat{k}$

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$