The Chain Rule

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A special rule, **the chain rule**, exists for differentiating a function of another function. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain what is meant by a function of a function
- state the chain rule
- differentiate a function of a function

Contents

1. Introduction

In this unit we learn how to differentiate a 'function of a function'. We first explain what is meant by this term and then learn about the Chain Rule which is the technique used to perform the differentiation.

2. A function of a function

Consider the expression $\cos x^2$. Immediately we note that this is different from the straightforward cosine function, $\cos x$. We are finding the cosine of x^2, not simply the cosine of x. We call such an expression a 'function of a function'.

Suppose, in general, that we have two functions, $f(x)$ and $g(x)$. Then

$$y = f(g(x))$$

is a function of a function. In our case, the function f is the cosine function and the function g is the square function. We could identify them more mathematically by saying that

$$f(x) = \cos x \qquad g(x) = x^2$$

so that

$$f(g(x)) = f(x^2) = \cos x^2$$

Now let's have a look at another example. Suppose this time that f is the square function and g is the cosine function. That is,

$$f(x) = x^2 \qquad g(x) = \cos x$$

then

$$f(g(x)) = f(\cos x) = (\cos x)^2$$

We often write $(\cos x)^2$ as $\cos^2 x$. So $\cos^2 x$ is also a function of a function.

In the following section we learn how to differentiate such a function.

3. The chain rule

In order to differentiate a function of a function, $y = f(g(x))$, that is to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, we need to do two things:

1. Substitute $u = g(x)$. This gives us

$$y = f(u)$$

Next we need to use a formula that is known as the Chain Rule.

2. **Chain Rule**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Key Point

Chain rule:

To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example

Suppose we want to differentiate $y = \cos x^2$.

Let $u = x^2$ so that $y = \cos u$.

It follows immediately that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = -\sin u$$

The chain rule says

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

and so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin u \times 2x \\ &= -2x \sin x^2 \end{aligned}$$

Example

Suppose we want to differentiate $y = \cos^2 x = (\cos x)^2$.

Let $u = \cos x$ so that $y = u^2$

It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 2u$$

Then

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 2u \times -\sin x \\ &= -2 \cos x \sin x \end{aligned}$$

Example

Suppose we wish to differentiate $y = (2x - 5)^{10}$.

Now it might be tempting to say 'surely we could just multiply out the brackets'. To multiply out the brackets would take a long time and there are lots of opportunities for making mistakes. So let us treat this as a function of a function.

Let $u = 2x - 5$ so that $y = u^{10}$. It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 10u^9$$

Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 10u^9 \times 2 \\ &= 20(2x-5)^9\end{aligned}$$

4. Some examples involving trigonometric functions

In this section we consider a trigonometric example and develop it further to a more general case.

Example

Suppose we wish to differentiate $y = \sin 5x$.

Let $u = 5x$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

From the chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \cos u \times 5 \\ &= 5\cos 5x\end{aligned}$$

Notice how the 5 has appeared at the front, - and it does so because the derivative of $5x$ was 5. So the question is, could we do this with any number that appeared in front of the x, be it 5 or 6 or $\frac{1}{2}$, 0.5 or for that matter n ?

So let's have a look at another example.

Example

Suppose we want to differentiate $y = \sin nx$.

Let $u = nx$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = n \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

Quoting the formula again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos u \times n \\ &= n\cos nx\end{aligned}$$

So the n's have behaved in exactly the same way that the 5's behaved in the previous example.

Key Point

if $y = \sin nx$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = n\cos nx$

For example, suppose $y = \sin 6x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6\cos 6x$ just by using the standard result.

Similar results follow by differentiating the cosine function:

Key Point

if $y = \cos nx$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -n\sin nx$

So, for example, if $y = \cos \frac{1}{2}x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\sin\frac{1}{2}x$.

5. A simple technique for differentiating directly

In this section we develop, through examples, a further result.

Example

Suppose we want to differentiate $y = \mathrm{e}^{x^3}$.

Let $u = x^3$ so that $y = \mathrm{e}^u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u$$

Quoting the formula again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^u \times 3x^2 \\ &= 3x^2\mathrm{e}^{x^3}\end{aligned}$$

We will now explore how this relates to a general case, that of differentiating $y = f(g(x))$.
To differentiate $y = f(g(x))$, we let $u = g(x)$ so that $y = f(u)$.

The chain rule states

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

In what follows it will be convenient to reverse the order of the terms on the right:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}u}$$

which, in terms of f and g we can write as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d}{dx}(g(x)) \times \frac{d}{du}(f(g((x)))$$

This gives us a simple technique which, with some practice, enables us to apply the chain rule directly

Key Point

(i) given $y = f(g(x))$, identify the functions $f(u)$ and $g(x)$ where $u = g(x)$.

(ii) differentiate g and multiply by the derivative of f
where it is understood that the argument of f is $u = g(x)$.

Example

To differentiate $y = \tan x^2$ we apply these two stages:

(i) first identify $f(u)$ and $g(x)$: $f(u) = \tan u$ and $g(x) = x^2$.

(ii) differentiate $g(x)$: $\dfrac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is $\sec^2 u$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\sec^2 x^2$$

Example

To differentiate $y = \mathrm{e}^{1+x^2}$.

(i) first identify $f(u)$ and $g(x)$: $f(u) = \mathrm{e}^u$ and $g(x) = 1 + x^2$.

(ii) differentiate $g(x)$: $\dfrac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is e^u to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\,\mathrm{e}^{1+x^2}$$

You should be able to verify the remaining examples purely by inspection. Try it!

Example

$$\begin{aligned} y &= \sin(x + \mathrm{e}^x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (1 + \mathrm{e}^x)\cos(x + \mathrm{e}^x) \end{aligned}$$

Example

$$\begin{aligned} y &= \tan(x^2 + \sin x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x + \cos x).\sec^2(x^2 + \sin x) \end{aligned}$$

Example

$$\begin{aligned} y &= (2 - x^5)^9 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -5x^4.\ 9(2 - x^5)^8 \\ &= -45x^4(2 - x^5)^8 \end{aligned}$$

Example

$$\begin{aligned} y &= \ln(x + \sin x) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (1 + \cos x)\ .\ \frac{1}{x + \sin x} \\ &= \frac{1 + \cos x}{x + \sin x} \end{aligned}$$

Exercises

1. Find the derivative of each of the following:
   a) $(3x - 7)^{12}$ b) $\sin(5x + 2)$ c) $\ln(2x - 1)$ d) e^{2-3x}
   e) $\sqrt{5x - 3}$ f) $(6x + 5)^{5/3}$ g) $\dfrac{1}{(3 - x)^4}$ h) $\cos(1 - 4x)$
2. Find the derivative of each of the following:
   a) $\ln(\sin x)$ b) $\sin(\ln x)$ c) $\mathrm{e}^{-\cos x}$ d) $\cos(\mathrm{e}^{-x})$
   e) $(\sin x + \cos x)^3$ f) $\sqrt{1 + x^2}$ g) $\dfrac{1}{\cos x}$ h) $\dfrac{1}{x^2 + 2x + 1}$

3. Find the derivative of each of the following:

a) $\ln\left(\sin^2 x\right)$ b) $\sin^2(\ln x)$ c) $\sqrt{\cos(3x-1)}$ d) $\left[1+\cos\left(x^2-1\right)\right]^{3/2}$

Answers

1. a) $36(3x-7)^{11}$ b) $5\cos(5x+2)$ c) $\dfrac{2}{2x-1}$ d) $-3e^{2-3x}$

e) $\dfrac{5}{2\sqrt{5x-3}}$ f) $10(6x+5)^{2/3}$ g) $\dfrac{4}{(3-x)^5}$ h) $4\sin(1-4x)$

2. a) $\dfrac{\cos x}{\sin x} = \cot x$ b) $\dfrac{\cos(\ln x)}{x}$ c) $\sin x e^{-\cos x}$

d) $e^{-x}\sin\left(e^{-x}\right)$ e) $3(\cos x - \sin x)(\sin x + \cos x)^2$ f) $\dfrac{x}{\sqrt{1+x^2}}$

g) $\dfrac{\sin x}{\cos^2 x} = \tan x \sec x$ h) $\dfrac{-2(x+1)}{(x^2+2x+1)^4} = \dfrac{-2}{(x+1)^3}$

3. a) $\dfrac{2\cos x}{\sin x} = 2\cot x$ b) $\dfrac{2\sin(\ln x)\cos(\ln x)}{x}$

c) $\dfrac{-3\sin(3x-1)}{2\sqrt{\cos(3x-1)}}$ d) $-3x\sin\left(x^2-1\right)\left[1+\cos\left(x^2-1\right)\right]^{1/2}$

Double integrals

Notice: this material must not be used as a substitute for attending the lectures

0.1 What is a double integral?

Recall that a **single integral** is something of the form

$$\int_a^b f(x)\,dx$$

A **double integral** is something of the form

$$\iint_R f(x,y)\,dx\,dy$$

where R is called the **region of integration** and is a region in the (x,y) plane. The double integral gives us the volume under the surface $z = f(x,y)$, just as a single integral gives the area under a curve.

0.2 Evaluation of double integrals

To evaluate a double integral we do it in stages, starting from the inside and working out, using our knowledge of the methods for single integrals. The easiest kind of region R to work with is a rectangle. To evaluate

$$\iint_R f(x,y)\,dx\,dy$$

proceed as follows:

- work out the limits of integration if they are not already known
- work out the inner integral for a typical y
- work out the outer integral

0.3 Example

Evaluate

$$\int_{y=1}^{2}\int_{x=0}^{3}(1+8xy)\,dx\,dy$$

Solution. In this example the "inner integral" is $\int_{x=0}^{3}(1+8xy)\,dx$ with y treated as a constant.

$$\begin{aligned}
\text{integral} &= \int_{y=1}^{2}\left(\underbrace{\int_{x=0}^{3}(1+8xy)\,dx}_{\text{work out treating } y \text{ as constant}}\right)dy \\
&= \int_{y=1}^{2}\left[x+\frac{8x^2y}{2}\right]_{x=0}^{3}dy \\
&= \int_{y=1}^{2}(3+36y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \left[3y + \frac{36y^2}{2}\right]_{y=1}^{2} \\
&= (6+72) - (3+18) \\
&= 57
\end{aligned}
$$

0.4 Example

Evaluate

$$\int_0^{\pi/2}\int_0^1 y \sin x \, dy \, dx$$

Solution.

$$
\begin{aligned}
\text{integral} &= \int_0^{\pi/2}\left(\int_0^1 y \sin x \, dy\right) dx \\
&= \int_0^{\pi/2} \left[\frac{y^2}{2} \sin x\right]_{y=0}^{1} dx \\
&= \int_0^{\pi/2} \tfrac{1}{2} \sin x \, dx \\
&= \left[-\tfrac{1}{2}\cos x\right]_{x=0}^{\pi/2} = \tfrac{1}{2}
\end{aligned}
$$

0.5 Example

Find the volume of the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y) : \ \ 0 \le x \le 1 \ \ 0 \le y \le 2\}$.

Solution. The volume under any surface $z = f(x, y)$ and above a region R is given by

$$V = \iint_R f(x, y) \, dx \, dy$$

In our case

$$
\begin{aligned}
V &= \int_0^2\int_0^1 (4 - x - y) \, dx \, dy \\
&= \int_0^2 \left[4x - \tfrac{1}{2}x^2 - yx\right]_{x=0}^{1} dy = \int_0^2 (4 - \tfrac{1}{2} - y) \, dy \\
&= \left[\frac{7y}{2} - \frac{y^2}{2}\right]_{y=0}^{2} = (7-2) - (0) = 5
\end{aligned}
$$

The double integrals in the above examples are the easiest types to evaluate because they are examples in which all four limits of integration are constants. This happens when the region of integration is rectangular in shape. In non-rectangular regions of integration the limits are not all constant so we have to get used to dealing with non-constant limits. We do this in the next few examples.

0.6 Example

Evaluate

$$\int_0^2 \int_{x^2}^x y^2 x \, dy \, dx$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_0^2 \int_{x^2}^x y^2 x \, dy \, dx \\
&= \int_0^2 \left[\frac{y^3 x}{3} \right]_{y=x^2}^{y=x} dx \\
&= \int_0^2 \left(\frac{x^4}{3} - \frac{x^7}{3} \right) dx = \left[\frac{x^5}{15} - \frac{x^8}{24} \right]_0^2 \\
&= \frac{32}{15} - \frac{256}{24} = -\frac{128}{15}
\end{aligned}$$

0.7 Example

Evaluate

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos \frac{y}{x} \, dy \, dx$$

Solution. Recall from elementary calculus the integral $\int \cos my \, dy = \frac{1}{m} \sin my$ for m independent of y. Using this result,

$$\begin{aligned}
\text{integral} &= \int_{\pi/2}^{\pi} \left[\frac{1}{x} \frac{\sin \frac{y}{x}}{\frac{1}{x}} \right]_{y=0}^{y=x^2} dx \\
&= \int_{\pi/2}^{\pi} \sin x \, dx = [-\cos x]_{x=\pi/2}^{\pi} = 1
\end{aligned}$$

0.8 Example

Evaluate

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} \, dx \, dy$$

Solution.

$$\begin{aligned}
\text{integral} &= \int_1^4 \left[\frac{e^{x/\sqrt{y}}}{1/\sqrt{y}} \right]_{x=0}^{x=\sqrt{y}} dy \\
&= \int_1^4 (\sqrt{y} e - \sqrt{y}) \, dy = (e-1) \int_1^4 y^{1/2} \, dy \\
&= (e-1) \left[\frac{y^{3/2}}{3/2} \right]_{y=1}^4 = \frac{2}{3}(e-1)(8-1) \\
&= \frac{14}{3}(e-1)
\end{aligned}$$

0.9 Evaluating the limits of integration

When evaluating double integrals it is very common **not** to be told the limits of integration but simply told that the integral is to be taken over a certain specified region R in the (x, y) plane. In this case you need to work out the limits of integration for yourself. Great care has to be taken in carrying out this task. The integration can in principle be done in two ways: (i) integrating first with respect to x and then with respect to y, or (ii) first with respect to y and then with respect to x. The limits of integration in the two approaches will in general be quite different, but both approaches must yield the same answer. Sometimes one way round is considerably harder than the other, and in some integrals one way works fine while the other leads to an integral that cannot be evaluated using the simple methods you have been taught. There are no simple rules for deciding which order to do the integration in.

0.10 Example

Evaluate

$$\iint_D (3 - x - y)\, dA \qquad [dA \text{ means } dxdy \text{ or } dydx]$$

where D is the triangle in the (x, y) plane bounded by the x-axis and the lines $y = x$ and $x = 1$.

Solution. A good diagram is essential.

Method 1 : do the integration with respect to x first. In this approach we select a typical y value which is (for the moment) considered fixed, and we draw a **horizontal** line across the region D; this horizontal line intersects the y axis at the typical y value. Find out the values of x (they will depend on y) where the horizontal line **enters** and **leaves** the region D (in this problem it enters at $x = y$ and leaves at $x = 1$). These values of x will be the limits of integration for the inner integral. Then you determine what values y has to range between so that the horizontal line sweeps the entire region D (in this case y has to go from 0 to 1). This determines the limits of integration for the outer integral, the integral with respect to y. For this particular problem the integral becomes

$$\begin{aligned}
\iint_D (3 - x - y)\, dA &= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy \\
&= \int_0^1 \left[3x - \frac{x^2}{2} - yx \right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(\left(3 - \tfrac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right) \right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2 \right) dy = \left[\frac{5y}{2} - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} \\
&= \frac{5}{2} - 2 + \frac{1}{2} = 1
\end{aligned}$$

Method 2 : do the integration with respect to y first and then x. In this approach we select a "typical x" and draw a vertical line across the region D at that value of x.

Vertical line enters D at $y = 0$ and leaves at $y = x$. We then need to let x go from 0 to 1 so that the vertical line sweeps the entire region. The integral becomes

$$\begin{aligned}
\iint_D (3 - x - y)\, dA &= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx \\
&= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\
&= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx \\
&= \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{1} = 1
\end{aligned}$$

Note that Methods 1 and 2 give the same answer. If they don't it means something is wrong.

0.11 Example

Evaluate

$$\iint_D (4x + 2)\, dA$$

where D is the region enclosed by the curves $y = x^2$ and $y = 2x$.

Solution. Again we will carry out the integration both ways, x first then y, and then vice versa, to ensure the same answer is obtained by both methods.

Method 1 : We do the integration first with respect to x and then with respect to y. We shall need to know where the two curves $y = x^2$ and $y = 2x$ intersect. They intersect when $x^2 = 2x$, i.e. when $x = 0, 2$. So they intersect at the points $(0, 0)$ and $(2, 4)$.

For a typical y, the horizontal line will enter D at $x = y/2$ and leave at $x = \sqrt{y}$. Then we need to let y go from 0 to 4 so that the horizontal line sweeps the entire region. Thus

$$\begin{aligned}
\iint_D (4x + 2)\, dA &= \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} (4x + 2)\, dx\, dy \\
&= \int_0^4 \left[2x^2 + 2x \right]_{x=y/2}^{x=\sqrt{y}} dy = \int_0^4 \left((2y + 2\sqrt{y}) - \left(\frac{y^2}{2} + y \right) \right) dy \\
&= \int_0^4 \left(y + 2y^{1/2} - \frac{y^2}{2} \right) dy = \left[\frac{y^2}{2} + \frac{2y^{3/2}}{3/2} - \frac{y^3}{6} \right]_0^4 = 8
\end{aligned}$$

Method 2 : Integrate first with respect to y and then x, i.e. draw a vertical line across D at a typical x value. Such a line enters D at $y = x^2$ and leaves at $y = 2x$. The integral becomes

$$\begin{aligned}
\iint_D (4x+2)\, dA &= \int_0^2 \int_{x^2}^{2x} (4x+2)\, dy\, dx \\
&= \int_0^2 [4xy + 2y]_{y=x^2}^{y=2x}\, dx \\
&= \int_0^2 \left(\left(8x^2 + 4x\right) - \left(4x^3 + 2x^2\right)\right)\, dx \\
&= \int_0^2 (6x^2 - 4x^3 + 4x)\, dx = \left[2x^3 - x^4 + 2x^2\right]_0^2 = 8
\end{aligned}$$

The example we have just done shows that it is sometimes easier to do it one way than the other. The next example shows that sometimes the difference in effort is more considerable. There is no general rule saying that one way is always easier than the other; it depends on the individual integral.

0.12 Example

Evaluate

$$\iint_D (xy - y^3)\, dA$$

where D is the region consisting of the square $\{(x,y) : \ -1 \le x \le 0, \ 0 \le y \le 1\}$ together with the triangle $\{(x,y) : \ x \le y \le 1, \ 0 \le x \le 1\}$.

Method 1 : (easy). integrate with respect to x first. A diagram will show that x goes from -1 to y, and then y goes from 0 to 1. The integral becomes

$$\begin{aligned}
\iint_D (xy - y^3)\, dA &= \int_0^1 \int_{-1}^{y} (xy - y^3)\, dx\, dy \\
&= \int_0^1 \left[\frac{x^2}{2}y - xy^3\right]_{x=-1}^{x=y} dy \\
&= \int_0^1 \left(\left(\frac{y^3}{2} - y^4\right) - (\tfrac{1}{2}y + y^3)\right) dy \\
&= \int_0^1 \left(-\frac{y^3}{2} - y^4 - \tfrac{1}{2}y\right) dy = \left[-\frac{y^4}{8} - \frac{y^5}{5} - \frac{y^2}{4}\right]_{y=0}^{1} = -\frac{23}{40}
\end{aligned}$$

Method 2 : (harder). It is necessary to break the region of integration D into two sub-regions D_1 (the square part) and D_2 (triangular part). The integral over D is given by

$$\iint_D (xy - y^3)\, dA = \iint_{D_1} (xy - y^3)\, dA + \iint_{D_2} (xy - y^3)\, dA$$

which is the analogy of the formula $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ for single integrals. Thus

$$
\begin{aligned}
\iint_D (xy - y^3)\,dA &= \int_{-1}^0 \int_0^1 (xy - y^3)\,dy\,dx + \int_0^1 \int_x^1 (xy - y^3)\,dy\,dx \\
&= \int_{-1}^0 \left[\frac{xy^2}{2} - \frac{y^4}{4}\right]_{y=0}^1 dx + \int_0^1 \left[\frac{xy^2}{2} - \frac{y^4}{4}\right]_{y=x}^1 dx \\
&= \int_{-1}^0 \left(\tfrac{1}{2}x - \frac{1}{4}\right) dx + \int_0^1 \left(\left(\frac{x}{2} - \frac{1}{4}\right) - \left(\frac{x^3}{2} - \frac{x^4}{4}\right)\right) dx \\
&= \left[\frac{x^2}{4} - \frac{x}{4}\right]_{-1}^0 + \left[\frac{x^2}{4} - \frac{x}{4} - \frac{x^4}{8} + \frac{x^5}{20}\right]_0^1 \\
&= -\tfrac{1}{2} - \frac{3}{40} = -\frac{23}{40}
\end{aligned}
$$

In the next example the integration can only be done one way round.

0.13 Example

Evaluate

$$\iint_D \frac{\sin x}{x}\,dA$$

where D is the triangle $\{(x, y) :\ 0 \le y \le x,\ 0 \le x \le \pi\}$.

Solution. Let's try doing the integration first with respect to x and then y. This gives

$$\iint_D \frac{\sin x}{x}\,dA = \int_0^\pi \int_y^\pi \frac{\sin x}{x}\,dx\,dy$$

but we cannot proceed because we cannot find an indefinite integral for $\sin x/x$. So, let's try doing it the other way. We then have

$$
\begin{aligned}
\iint_D \frac{\sin x}{x}\,dA &= \int_0^\pi \int_0^x \frac{\sin x}{x}\,dy\,dx \\
&= \int_0^\pi \left[\frac{\sin x}{x}\,y\right]_{y=0}^x dx = \int_0^\pi \sin x\,dx \\
&= [-\cos x]_0^\pi = 1 - (-1) = 2
\end{aligned}
$$

0.14 Example

Find the volume of the tetrahedron that lies in the first octant and is bounded by the three coordinate planes and the plane $z = 5 - 2x - y$.

Solution. The given plane intersects the coordinate axes at the points $(\frac{5}{2}, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 5)$. Thus, we need to work out the double integral

$$\iint_D (5 - 2x - y)\,dA$$

where D is the triangle in the (x, y) plane with vertices $(x, y) = (0, 0)$, $(\frac{5}{2}, 0)$ and $(0, 5)$. It is a good idea to draw another diagram at this stage showing just the region D in the (x, y) plane. Note that the equation of the line joining the points $(\frac{5}{2}, 0)$ and $(0, 5)$ is $y = -2x + 5$. Then:

$$\begin{aligned}
\text{volume} &= \iint_D (5 - 2x - y)\, dA = \int_0^5 \int_0^{(5-y)/2} (5 - 2x - y)\, dx\, dy \\
&= \int_0^5 \Big[5x - x^2 - yx\Big]_{x=0}^{x=(5-y)/2} dy \\
&= \int_0^5 \left[5\left(\frac{5-y}{2}\right) - \left(\frac{5-y}{2}\right)^2 - y\left(\frac{5-y}{2}\right)\right] dy \\
&= \int_0^5 \left(\frac{25}{4} - \frac{5y}{2} + \frac{y^2}{4}\right) dy \\
&= \left[\frac{25y}{4} - \frac{5y^2}{4} + \frac{y^3}{12}\right]_0^5 = \frac{125}{12}
\end{aligned}$$

0.15 Changing variables in a double integral

We know how to change variables in a **single** integral:

$$\int_a^b f(x)\, dx = \int_A^B f(x(u)) \frac{dx}{du}\, du$$

where A and B are the new limits of integration.
For **double integrals** the rule is more complicated. Suppose we have

$$\iint_D f(x, y)\, dx\, dy$$

and want to change the variables to u and v given by $x = x(u, v)$, $y = y(u, v)$. The change of variables formula is

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v))|J|\, du\, dv \qquad (0.1)$$

where J is the Jacobian, given by

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

and D^* is the new region of integration, in the (u, v) plane.

0.16 Transforming a double integral into polars

A very commonly used substitution is conversion into polars. This substitution is particularly suitable when the region of integration D is a circle or an annulus (i.e. region between two concentric circles). Polar coordinates r and θ are defined by

$$x = r\cos\theta, \quad y = r\sin\theta$$

The variables u and v in the general description above are r and θ in the polar coordinates context and the Jacobian for polar coordinates is

$$\begin{aligned} J &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

So $|J| = r$ and the change of variables rule (0.1) becomes

$$\iint_D f(x,y)\, dx\, dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

0.17 Example

Use polar coordinates to evaluate

$$\iint_D xy\, dx\, dy$$

where D is the portion of the circle centre 0, radius 1, that lies in the first quadrant.
Solution. For the portion in the first quadrant we need $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. These inequalities give us the limits of integration in the r and θ variables, and these limits will all be constants.
With $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\begin{aligned} \iint_D xy\, dx\, dy &= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\, r\, dr\, d\theta \\ &= \int_0^{\pi/2}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^1 d\theta \\ &= \int_0^{\pi/2}\frac{1}{4}\sin\theta\cos\theta\, d\theta = \int_0^{\pi/2}\frac{1}{8}\sin 2\theta\, d\theta \\ &= \frac{1}{8}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{8} \end{aligned}$$

0.18 Example

Evaluate

$$\iint_D e^{-(x^2+y^2)}\, dx\, dy$$

where D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Solution. It is not feasible to attempt this integral by any method other than transforming into polars.
Let $x = r\cos\theta$, $y = r\sin\theta$. In terms of r and θ the region D between the two circles is described by $1 \le r \le 2$, $0 \le \theta \le 2\pi$, and so the integral becomes

$$\iint_D e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_1^2 e^{-r^2}\, r\, dr\, d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left[-\tfrac{1}{2}e^{-r^2}\right]_{r=1}^{2} d\theta \\
&= \int_0^{2\pi} \left(-\tfrac{1}{2}e^{-4} + \tfrac{1}{2}e^{-1}\right) d\theta \\
&= \pi(e^{-1} - e^{-4})
\end{aligned}
$$

0.19 Example: integrating e^{-x^2}

The function e^{-x^2} has no elementary antiderivative. But we can evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ by using the theory of double integrals.

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty} e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}
$$

Now transform to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The region of integration is the whole (x, y) plane. In polar variables this is given by $0 \le r < \infty$, $0 \le \theta \le 2\pi$. Thus

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \left[-\tfrac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta \\
&= \int_0^{2\pi} \tfrac{1}{2}d\theta = \pi
\end{aligned}
$$

We have shown that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

The above integral is very important in numerous applications.

0.20 Other substitutions

So far we have only illustrated how to convert a double integral into polars. We will now illustrate some examples of double integrals that can be evaluated by other substitutions. Unlike single integrals, for a double integral the choice of substitution is often dictated not only by what we have in the integrand but also by the shape of the region of integration.

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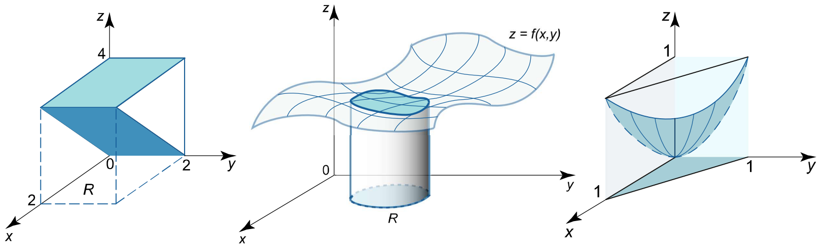

Physical Applications of Double Integrals

Mass and Static Moments of a Lamina

Suppose we have a lamina which occupies a region R in the xy-plane and is made of non-homogeneous material. Its density at a point (x, y) in the region R is $\rho(x, y)$. The total mass of the lamina is expressed through the double integral as follows:

$$m = \iint\limits_R \rho(x, y)\, dA.$$

The static moment of the lamina about the x-axis is given by the formula

$$M_x = \iint\limits_R y\rho(x,y)\,dA.$$

Similarly, the static moment of the lamina about the y-axis is

$$M_y = \iint\limits_R x\rho(x,y)\,dA.$$

The coordinates of the center of mass of a lamina occupying the region R in the xy-plane with density function $\rho(x,y)$ are described by the formulas

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint\limits_R x\rho(x,y)\,dA = \frac{\iint\limits_R x\rho(x,y)\,dA}{\iint\limits_R \rho(x,y)\,dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint\limits_R y\rho(x,y)\,dA = \frac{\iint\limits_R y\rho(x,y)\,dA}{\iint\limits_R \rho(x,y)\,dA}.$$

When the mass density of the lamina is $\rho(x,y) = 1$ for all (x,y) in the region R, the center of mass is defined only by the shape of the region and is called the centroid of R.

Moments of Inertia of a Lamina

The moment of inertia of a lamina about the x-axis is defined by the formula

$$I_x = \iint\limits_R y^2\rho(x,y)\,dA.$$

Similarly, the moment of inertia of a lamina about the y-axis is given by

$$I_y = \iint\limits_R x^2\rho(x,y)\,dA.$$

The polar moment of inertia is

$$I_0 = \iint\limits_R \left(x^2 + y^2\right)\rho(x,y)\,dA.$$

Charge of a Plate

Suppose electrical charge is distributed over a region which has area R in the xy-plane and its charge density is defined by the function $\sigma(x, y)$. Then the total charge Q of the plate is defined by the expression

$$Q = \iint\limits_R \sigma(x, y)\, dA.$$

Average of a Function

We give here the formula for calculation of the average value of a distributed function. Let $f(x, y)$ be a continuous function over a closed region R in the xy-plane. The average value μ of the function $f(x, y)$ in the region R is given by the formula

$$\mu = \frac{1}{S} \iint\limits_R f(x, y)\, dA,$$

where $S = \iint\limits_R dA$ is the area of the region of integration R.

Solved Problems

Click or tap a problem to see the solution.

Example 1

Find the centroid of the lamina cut by the parabolas $y^2 = x$ and $y = x^2$.

Example 2

Calculate the moments of inertia of the triangle bounded by the straight lines $x + y = 1$, $x = 0$, $y = 0$ (Figure 2) and having density $\rho(x, y) = xy$.

Example 3

Electric charge is distributed over the disk $x^2 + y^2 = 1$ so that its charge density is $\sigma(x, y) = 1 + x^2 + y^2 \left(\text{Kl/m}^2\right)$. Calculate the total charge of the disk.

Example 1.

Find the centroid of the lamina cut by the parabolas $y^2 = x$ and $y = x^2$.

Solution.

The lamina has the form shown in Figure 1.

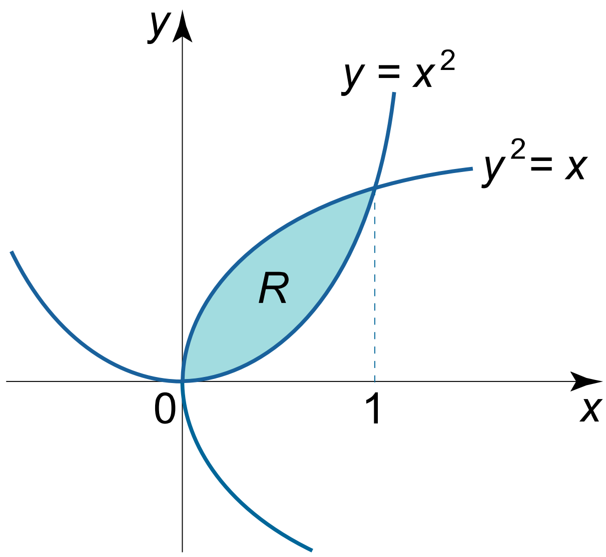

Figure 1.

Since it is homogeneous, we suppose that the density $\rho(x,y) = 1$. The mass of the lamina is

$$m = \iint\limits_R dA = \int\limits_0^1 \left[\int\limits_{x^2}^{\sqrt{x}} dy \right] dx = \int\limits_0^1 \left[y|_{x^2}^{\sqrt{x}} \right] dx = \int\limits_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= \int\limits_0^1 \left(x^{\frac{1}{2}} - x^2\right) dx = \left(\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Now we find the moment of the lamina about the x-axis and y-axis.

$$M_x = \iint\limits_R y dA = \int\limits_0^1 \left[\int\limits_{x^2}^{\sqrt{x}} y dy \right] dx = \int\limits_0^1 \left[\left(\frac{y^2}{2}\right)\Bigg|_{x^2}^{\sqrt{x}} \right] dx = \frac{1}{2} \int\limits_0^1 \left(x - x^4\right) dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{20},$$

$$M_y = \iint\limits_R x dA = \int\limits_0^1 \left[\int\limits_{x^2}^{\sqrt{x}} dy \right] x dx = \int\limits_0^1 \left(\sqrt{x} - x^2\right) x dx = \int\limits_0^1 \left(x^{\frac{3}{2}} - x^3\right) dx$$

$$= \left(\frac{2x^{\frac{5}{2}}}{5} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}.$$

Thus, the coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{3}{20}}{\frac{1}{3}} = \frac{9}{20}, \ \bar{y} = \frac{M_x}{m} = \frac{\frac{3}{20}}{\frac{1}{3}} = \frac{9}{20}.$$

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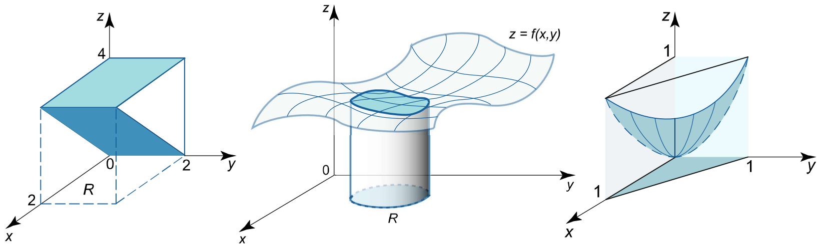

Physical Applications of Double Integrals – Page 2

Example 2.

Calculate the moments of inertia of the triangle bounded by the straight lines $x + y = 1$, $x = 0$, $y = 0$ (Figure 2) and having density $\rho(x, y) = xy$.

Solution.

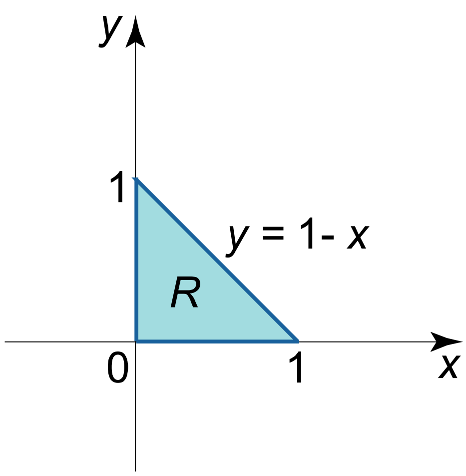

Figure 2.

The moment of inertia about the x-axis is

$$I_x = \iint_R y^2\rho(x,y)\,dxdy = \int_0^1 \left[\int_0^{1-x} y^2xydy\right]dx = \int_0^1 \left[\int_0^{1-x} y^3dy\right]xdx$$

$$= \int_0^1 [(\frac{y^4}{4})\Big|_0^{1-x}]\,xdx = \frac{1}{4}\int_0^1 (1-x)^4xdx = \frac{1}{4}\int_0^1 (1-4x+6x^2-4x^3+x^4)\,xdx$$

$$= \frac{1}{4}\int_0^1 (x-4x^2+6x^3-4x^4+x^5)\,dx = \frac{1}{4}(\frac{x^2}{2} - \frac{4x^3}{3} + \frac{6x^4}{4} - \frac{4x^5}{5} + \frac{x^6}{6})\Big|_0^1$$

$$= \frac{1}{4}(\frac{1}{2} - \frac{4}{3} + \frac{3}{2} - \frac{4}{5} + \frac{1}{6}) = \frac{1}{120}.$$

Similarly, we can find the moment of inertia about the y-axis:

$$I_y = \iint_R x^2\rho(x,y)\,dxdy = \int_0^1 \left[\int_0^{1-x} x^2xydy\right]dx = \int_0^1 \left[\int_0^{1-x} ydy\right]x^3dx$$

$$= \int_0^1 [(\frac{y^2}{2})\Big|_0^{1-x}]\,x^3dx = \frac{1}{2}\int_0^1 (1-x)^2x^3dx = \frac{1}{2}\int_0^1 (1-2x+x^2)\,x^3dx$$

$$= \frac{1}{2}\int_0^1 (x^3-2x^4+x^5)\,dx = \frac{1}{2}(\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6})\Big|_0^1 = \frac{1}{2}(\frac{1}{4} - \frac{2}{5} + \frac{1}{6})$$

$$= \frac{1}{120}.$$

Example 3.

Electric charge is distributed over the disk $x^2 + y^2 = 1$ so that its charge density is $\sigma(x,y) = 1 + x^2 + y^2$ (Kl/m^2). Calculate the total charge of the disk.

Solution.

In polar coordinates, the region occupied by the disk is defined by the set $\{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$. The total charge is

$$Q = \iint_R \sigma(x,y)\,dxdy = \int_0^{2\pi} d\theta \int_0^1 (1+r^2)\,rdr = 2\pi\int_0^1 (r+r^3)\,dr = 2\pi(\frac{r^2}{2} + \frac{r^4}{4})\Big|_0^1$$

$$= 2\pi(\frac{1}{2} + \frac{1}{4}) = \frac{3\pi}{2}\ (Kl).$$

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Contact

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Tools

Example

Find the first moment about the y-axis of a thin plate of uniform density δ bounded by $x=0$ (1), $y=x$ (2), and $y=2-x^2$ (3) ?

$dy\,dx$ $dy\,dx$

Solution

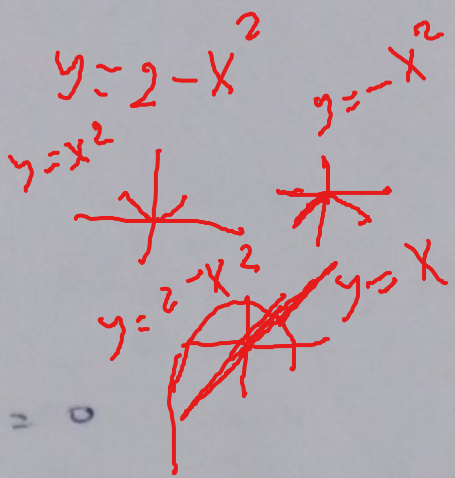
$x = 2-x^2$

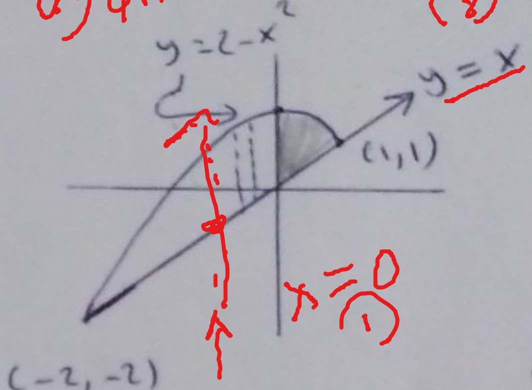
$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2 \Rightarrow y=-2$

$x=1 \Rightarrow y=1$





3 boundaries

$$M_y = \int_0^1 \int_x^{2-x^2} x\,\delta \; dy\,dx$$

$$= \int_0^1 \delta x\,dx\,[y]_x^{2-x^2} = \int_0^1 \delta x\,dx\,(2-x^2-x)$$

$$= \int_0^1 \delta\,(2x-x^3-x^2)\,dx = \delta\left[x^2-\frac{x^4}{4}-\frac{x^3}{3}\right]_0^1$$

$$= \delta\,\frac{5}{12}$$

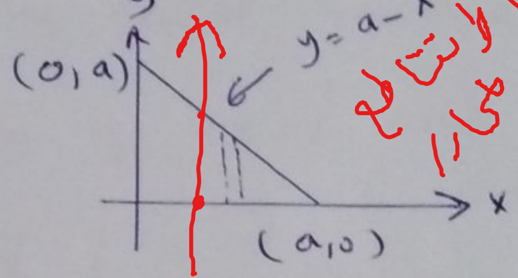
(24)

Example: Find the centroid of the triangular region cut from the first quadrant by the line $x+y=a$, $a>0$?

يطلب $(\bar{x},\bar{y})$ ؟

Note: $\delta = 1$

$dy\,dx$



Solution:

$$A=\int_0^a\int_0^{a-x} dy\,dx$$

$y=a-x$

$y=x$

$$=\int_0^a y\Big]_0^{a-x} dx=\int_0^a (a-x)\,dx$$

$$=\left[ax-\frac{x^2}{2}\right]_0^a=\frac{a^2}{2}$$

$$\bar{x}=\frac{M_y}{A}=\frac{M_x}{A}$$

$y=a-x$

$x-y+a=0$

$y=x+a$

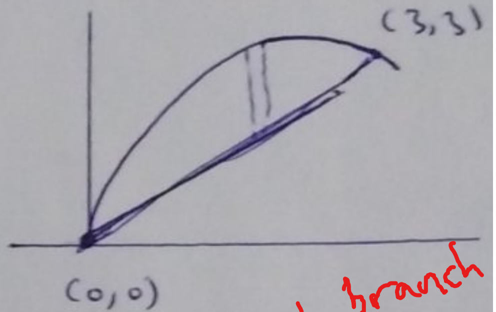
$$M_y=\int_0^a\int_0^{a-x} x\,dy\,dx \qquad =\int_0^a xy\Big]_0^{a-x} dx$$

$$=\int_0^a x(a-x)\,dx=\int_0^a (ax-x^2)\,dx$$

$$=\left[\frac{ax^2}{2}-\frac{x^3}{3}\right]_0^a=\frac{a^3}{6}$$

$$\bar{x} = \bar{y} = \frac{a^3/6}{a^2/2} = \frac{a}{3}$$

Example: The area of the region in the first quadrant bounded by $y = 4x - x^2$ and $y = x$ is $9/2$. Find $\bar{x}$?



Solution:

$$\bar{x} = \frac{M_y}{A}$$

$$4x - x^2 = x$$

$$x^2 - 3x = 0 \rightarrow x(x-3) = 0$$

$$x = 0, \quad y = 0$$

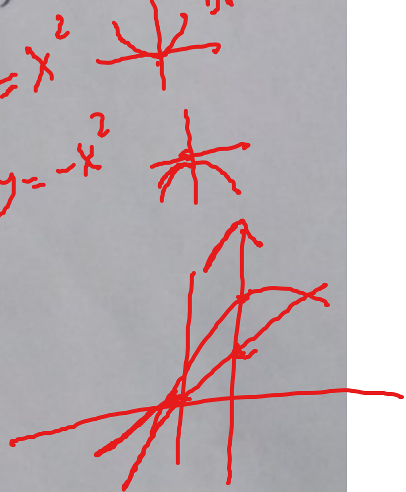
$$x = 3, \quad y = 3$$

branch

i) $y = x^2$

ii) $y = -x^2$

iii



$$M_y = \int_0^3 \int_x^{4x-x^2} x \, dy \, dx = \int_0^3 [y]_x^{4x-x^2} \, x \, dx$$

(24)

$$= \int_0^3 (4x^2 - x^3 - x^2)\,dx = \int_0^3 (3x^2 - x^3)\,dx$$

$$= \left[x^3 - \frac{x^4}{4}\right]_0^3 = \left[27 - \frac{51}{4}\right]$$

$$\bar{x} = \frac{M_y}{9/2}$$

5- Moment of intertia (w.r.t x-axis) is:

$$I_x = \iint \delta(x,y)\, dA\, y^2$$

6) Moment of intertia (w.r.t. y-axis) is:

$$I_y = \iint \delta(x,y)\, dA\, x^2 .$$

I_x to be found!

Example: Find the moment of Inertia with respect to the x-axis of a thin plate bounded by the parabola $x=y-y^2$ and the line $x+y=0$ if $\delta(x,y)=x+y$?

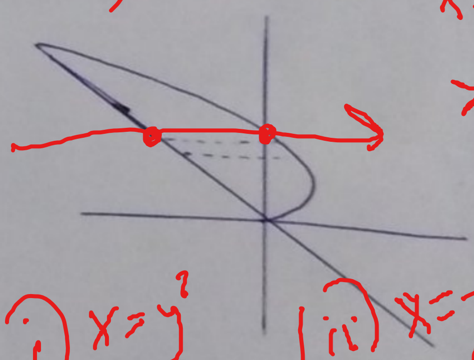
dx dy

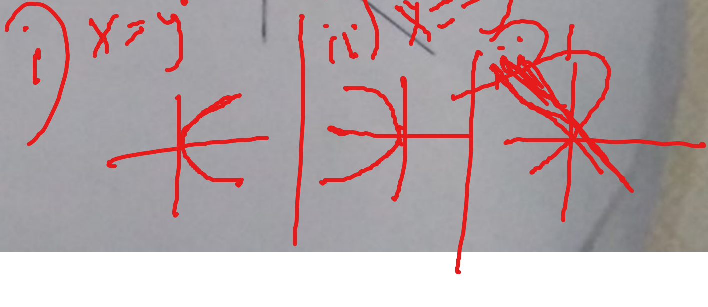
$x=-y$

Solution

$x=y-y^2$ and $x=-y$

$y-y^2=-y$





(18)

$y^2 - 2y = 0 \Rightarrow y(y-2) = 0$

$y = 0 \Rightarrow x = 0$

$y = 2 \Rightarrow x = -2$

$$I = \int_0^2 \int_{-y}^{y-y^2} y^2 \delta \, dx \, dy$$

$$= \int_0^2 \int_{-y}^{y-y^2} (x+y) y^2 \, dx \, dy$$

$$= \int_0^2 \left[\frac{x^2}{2} + xy \right]_{-y}^{y-y^2} y^2 \, dy = \int_0^2 \left[\frac{(y-y^2)^2}{2} + y(y-y^2) - \frac{y^2}{2} + y^2 \right] y^2 \, dy$$

$$= \int_0^2 \left[\frac{y^2}{2} - y^3 + \frac{y^4}{2} + y^2 - y^3 - \frac{y^2}{2} + y^2 \right] y^2 \, dy$$

$$= \int_0^2 \left[2y^4 - 2y^5 + \frac{y^6}{2} \right] dy$$

$$= \left[\frac{2}{5} y^5 - \frac{2}{6} y^6 + \frac{1}{14} y^7 \right]_0^2 = 0$$

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Triple Integrals

Ex: Evaluate the following triple integrals

1) $\int_0^2 \int_0^1 \int_0^{\pi} x e^y \sin z \, dz \, dy . dx$

$= \int_0^2 \int_0^1 x e^y [-\cos z] \, dy \, dx$

$= 2 \int_0^2 \int_0^1 x e^y \, dy \, dx = 2 \int_0^2 x e^y \Big|_0^1 dx = 2 \int_0^2 x(e-1) \, dx$

$= 2(e-1) \int_0^2 x \, dx = 4(e-1)$

2) $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} \, dx \, dy \, dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \ln x \Big|_1^{e^3} dy \, dz$

$= \int_1^e \int_1^{e^2} \frac{\ln e^3 - \ln 1}{yz} \, dy \, dz$

(37)

$$= 3 \int_1^e \int_1^{e^2} \frac{1}{yz}\, dy\, dz$$

$$= 3 \int_1^e \frac{1}{z} \ln y \Big|_1^{e^2} dz = 6 \int_1^e \frac{dz}{z} = 6 \ln|z| \Big|_1^e = 6$$

3) $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx = \int_0^1 \int_0^1 (x^2 + y^2 + \frac{z^3}{3}) \Big|_0^1 dy\, dx$

$$= \int_0^1 \int_0^1 (x^2 + y^2 + \frac{1}{3})\, dy\, dx = \int_0^1 (x^2 + \frac{2}{3})\, dx$$

$$= \frac{x^3}{3} + \frac{2}{3} x \Big|_0^1 = 1$$

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$$\int_{\frac{\pi}{2}}^{\frac{\pi}{3}}\int_{\frac{\pi}{4}}^{\pi}\int_{0}^{\frac{\pi}{2}} \sin\theta \cos 2\alpha \sin\gamma \, d\gamma \, d\alpha \, d\theta$$

2- $$\int_{1}^{2}\int_{2}^{3}\int_{1}^{3} \frac{1}{x^2 y z^3} \, dz \, dy \, dx$$

3- $$\int_{2}^{3}\int_{1}^{2}\int_{0}^{4} x^4 e^{2y} \sqrt{z} \, dz \, dy \, dx$$

$$= \int_{2}^{3}\int_{1}^{2} x^4 e^{2y} \left[\int_{0}^{4} \sqrt{z} \, dz\right] dy \, dx$$

$$= \int_{2}^{3}\int_{1}^{2} x^4 e^{2y} \frac{2}{3} \cdot z^{3/2}\Big|_{0}^{4} dy \, dx$$

$$= \int_{2}^{3}\int_{1}^{2} x^4 e^{2y} \frac{2}{3} 2^{2\cdot\frac{3}{2}} dy \, dx$$

$$= \frac{16}{3} \cdot \frac{1}{2}\left[\int_{2}^{3} x^4 \int_{1}^{2} e^{2y} \cdot 2 \, dy \, dx\right]$$

$$\int_{2}^{3} e^{2y}\Big|_{1}^{2} dx$$

Applications of Triple Integrals.

The first thing we must deal with when having triple Integrals is to set the limits of integrations, which will follow the following steps:

$$\text{Volume} = \int\int\int dv$$

order

1) $$V = \int_{x_1=\text{lower}}^{x_2=\text{upper}} \int_{y_1=-f(x)}^{y_2=f(x)} \int_{z_1=f(x,y)}^{z_2=f(x,y)} dz\, dy\, dx$$

2) let $x = y = 0 \Rightarrow z_1 = a$ (lower) and $z_2 = b$ (upper)

3) let $z_1 = z_2 \Rightarrow y = \mp f(x)$

4) In step (3) set $y = 0 \Rightarrow x = \mp \text{number}$

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Example: Set boundaries the limits of integrations for the volume bounded by elleptic paraboloid $z_1 = x^2 + 9y^2$ and $z_2 = 18 - x^2 - 9y^2$?

$dz\,dy\,dx$

Solution:

let $x = y = 0 \Rightarrow$

1) $z_1 = 0 \equiv$ lower and $z_2 = 18 \equiv$ upper

inner integration

$z_1 = z_2$

2) $x^2 + 9y^2 = 18 - x^2 - 9y^2$

نقوم بنقل الحدود

$x^2 + 9y^2 - 18 + x^2 + 9y^2 = 0$

$\left[18y^2 = 18 - 2x^2\right] \times \frac{1}{18}$

$y^2 = 1 - \frac{x^2}{9} \Rightarrow$

$y = \mp\sqrt{1 - \frac{x^2}{9}}$ } حدود y

3) let $y = 0 \Rightarrow 1 - \frac{x^2}{9} = 0 \Rightarrow 9 - x^2 = 0$

$\Rightarrow x = \mp 3$ } حدود x

(38)

and hence

$$V = \int_{x=-3}^{x=3} \int_{y=-\sqrt{1-\frac{x^2}{9}}}^{y=+\sqrt{1-\frac{x^2}{9}}} \int_{z=x^2+9y^2}^{z=18-x^2-9y^2} dz\, dy\, dx$$

① dz ② y or x

Example: Find the volume of the wedge cut from the cylinder $x^2+y^2=1$ by the plane $z=y$ above and the plane $z=0$?

upper

lower

Solution:

$z=0$ lower and
$z=y \equiv$ upper } given

②

$$x^2 = 1-y^2 \Rightarrow x = \mp\sqrt{1-y^2}$$

$1-y^2=0 \Rightarrow 1=y^2$

$y^2=1-x^2$

$y=+\sqrt{1-x^2}$

$dz\, dy\, dx$

or

$dz\, dx\, dy$

③ let $x=0 \Rightarrow y=\mp 1$, but since it requires volume $\Rightarrow y=0$ and $y=1$

$$\therefore V = \int_{y=0}^{y=1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \left[\int_{0}^{z=y} dz \right] dx\, dy$$

(3)

$$\left[\int_0^y 1\,dz\right] = z\Big|_0^y$$

لو الكامل لا زوجي

$$V = \int_{y=0}^{1}\left[\int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} y\,dx\right] dy$$

$$yx\Big|_{-\sqrt{\ }}^{\sqrt{\ }}$$

$$= \int_{y=0}^{1} yx\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy = \int_0^1 y\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]$$

$$= \int_{y=0}^{1} 2y\sqrt{1-y^2}\,dy$$

$\int$ مشتقة . الدالة $= \dfrac{(\text{الدالة})^2}{2} + C$

$$= -\frac{2}{3}(1-y^2)^{3/2}\Big|_0^1 = -\frac{2}{3}\left[0-1\right] = \frac{2}{3}$$

$$= -\frac{2}{3}\left(1-1^2\right)^{\frac{3}{2}} - \left(-\frac{2}{3}(1)^{3/2}\right)$$

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H-W

1) Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$?

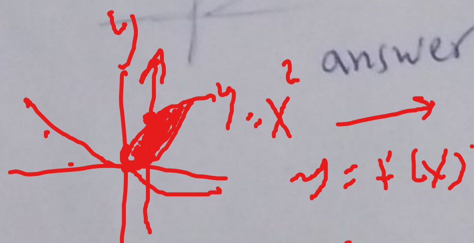
$\iiint dz\, dx\, dy$

2) Integrate the function $f(x,y,z) = xy$ over the volume enclosed by the planes $z = x + y$ and $z = 0$, and between the surfaces $y = x^2$ and $x = y^2$?

$y = \pm\sqrt{x}$

answer = $\frac{3}{28}$

$\iiint xy\, dz\, dy\, dx$



3) Evaluate the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$ and $9 \geq y^2 + z^2$?

$\iiint 1\, dz\, dy\, dx$

$\iiint z$

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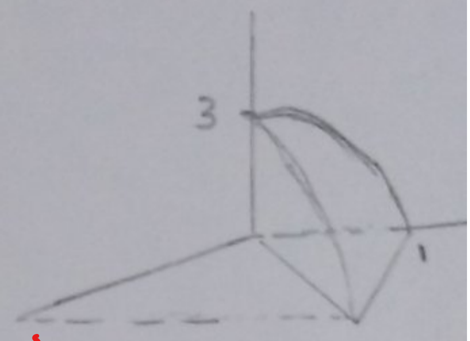
Solution 3



The integration limits are

- Limits in z

$$z \geq 0$$
$$z \leq \sqrt{9-y^2}$$

- $x \leq y/3$ and $x \geq 0$

- $y \geq 0$

$$y \leq 3$$

$$\therefore I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy$$

Main operators in vector calculus

① Gradiend partial derivative

② Divergence

③ Curl w.r.t $\frac{\partial}{\partial x}(xy^2) = y^2(1)$

$\frac{\partial}{\partial y} = x2y$

(x, y, z)

derivatives

$\frac{dy}{dx}$

Define the operator ∇ (del) by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

1) The Gradient (scalar to vector)

if we simply multiply a scalar field such as $p(x, y, z)$ by the del operator the result is a vector field, and the components of the vector at each point are just the partial derivatives of the scalar field at that point

$$\nabla P = i\frac{\partial P}{\partial x} + j\frac{\partial P}{\partial y} + k\frac{\partial P}{\partial z}$$

2) The Divergence (Scalar product, Dot product) (vector to Scalar)

The div of vector field

$$V(x,y,z) = V_x(x,y,z)i + V_y(x,y,z)j + V_z(x,y,z)k$$

is a scalar

$$\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Example :- find the div of

$$F(x,y) = x^2 y i + y x^2 j$$

sol :-

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (x^2 y, y x^2)$$

$$= 2xy + x^2$$

$f(x,y,z) = x\cos(xy) z i$

Example :- find the div of

$$F(x,y,z) = \cos z i + \sin y j + \tan x k$$

sol :-

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(\cos z i + \sin y j + \tan x k)$$

$$= 0 + \cos y + 0$$

$$= \cos y$$

xample :- find the div of

$$F(x,y,z) = 2xz\,i - xy\,j - z\,k$$

sol :-

$$\nabla \cdot F = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z)$$

$$= 2z - x - 1$$

The Curl (vector product, cross product)
(vector to vector)

The curl of a vector field

$$V(x,y,z) = V_x(x,y,z)\,i + V_y(x,y,z)\,j + V_z(x,y,z)\,k$$

is a vector :

$$\nabla \times V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (V_x, V_y, V_z)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix} i + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_x & V_z \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix} k$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) i + \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) j + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) k$$

Example :- find the curl of

$F = (x^2 - y)i + 4zj + x^2 k$

sol:-

$$\text{curl}\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z)\right)i - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2 - y)\right)j$$

$$+ \left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y)\right)k$$

$$= (0 - 4)i - (2x - 0)j + (0 + 1)k$$

$$= -4i - 2xj + k$$

Example :- if $F = x^2 i + y^3 j$ then calculate $\nabla \times F$

sol:-

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^3 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^3)\right)i - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right)j$$

$$+ \left(\frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial y}(x^2)\right)k$$

...

$= 0i - 0j + 0k = 0$

Example :- find the curl of

$$F = y^3 i + x^2 j$$

Sol :-

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & x^2 & 0 \end{vmatrix}$$

$$= (0)i - (0)j + (2x - 3y^2)k$$

$$= (2x - 3y^2)k$$

H.W

① if $F = 3xy^2 i + e^z j + xy \sin z k$ calculate $\nabla \cdot F$

② if $F = (y-x)i + (z-y)j + (x-z)k$ find $\nabla \cdot F$

③ if $F (x^2+y^2+z^2)i + (x^4 - y^2 z^2)j + xyz k$ find $\nabla \times F$

④ find the curl $(\nabla \times F)$ of $F = x^2 i - 2xyj + 3zxk$

Integrals of form $\int_c F \times dr$ ① (44) (22)

contour

$F = F_1 i + F_2 j + F_3 k$
$dr = dx\,i + dy\,j + dz\,k$

①

$$F \times dr = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix} = (F_2\,dz - F_3\,dy)i - (F_1\,dz - F_3\,dx)j + (F_1\,dy - F_2\,dx)k$$

$$= (F_3 j - F_2 k)dx + (F_1 k - F_3 i)dy + (F_2 i - F_1 j)dz$$

Example 1 :-

vector

Evaluate the integral $\int_c (x^2 i + 3xy\,j) \times dr$ cross where C represents the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$. $\Rightarrow dx \rightarrow x: 0, 1$ $\quad y = 2x^2$

$\frac{dy}{dx} = 4x \quad dy = 4x\,dx$

Sol :-

$$F \times dr = \begin{vmatrix} i & j & k \\ x^2 & 3xy & 0 \\ dx & dy & dz \end{vmatrix} = 0i - 0j + (x^2\,dy - 3xy\,dx)k$$

$$\int_c (x^2 i + 3xy\,j) \times dr = \int_c (x^2\,dy - 3xy\,dx)k$$

$y = 2x^2 \qquad dy = 4x\,dx$

...

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$$= \int_{x=0}^{1} (x^2 \underbrace{4x\,dx}_{dy} - 3x\, 2x^2 dx)\,k$$

$$= \int_0^1 (4x^3 dx - 6x^3 dx)\,k = \int_0^1 -2x^3 dx\, k$$

$$= -\frac{1}{2} x^4 \Big|_0^1 k$$

$$= -\frac{1}{2} k \leftarrow \text{vector}$$

i, j, k

② Integrals of form $\int_c (\nabla \cdot F)\,dr$

$F = F_1 i + F_2 j + F_3 k$

$dr = dx\,i + dy\,j + dz\,k$

Example:

Find the vector Line integral $\int_c (\nabla \cdot F)\,dr$ where $F = x^2 i + 2xyj + 2xzk$ and c is the curve $y = x^2$, $z = x^3$ from $x=0$ to $x=1$

cross $\times \rightarrow$ vector

$\cdot \rightarrow$ non-vector

Sol:-

$F = x^2 i + 2xyj + 2xzk$

$$\nabla \cdot F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(2xz)$$

$$= 2x + 2x + 2x = 6x$$

$$\int_c (\nabla \cdot F)\,dr = \int_c 6x\,(dx\,i + dy\,j + dz\,k)$$

$$= \int_c 6x\,dx\,i + \int_c 6x\,dy\,j + \int_c 6x\,dz\,k$$

(4b) (27)

The first term is

$$\int (6x\,dx)\,i = \int_{x=0}^{1} 6x\,dx\,i = 3x^2\Big|_0^1 i = 3i$$

The second term $y = x^2$ $\quad dy = 2x\,dx \quad \frac{dy}{dx} = 2x$

$$\int 6x\,dy\,j = \int_{x=0}^{1} (6x\,2x\,dx)\,j = \int_0^1 12x^2\,dx\,j = 4x^3\Big|_0^1 j = 4j$$

The third term $z = x^3$ $\quad dz = 3x^2\,dx$

$$\int_c 6x\,dz\,k = \int_{x=0}^{1} 6x\,3x^2\,dx\,k = \int_0^1 18x^3\,dx\,k = \frac{9}{2}x^4\Big|_0^1 k$$

$$= \frac{9}{2}k$$

$$\int_c (\nabla \cdot F)\,dr = 3i + 4j + \frac{9}{2}k$$

H.W:-

① Evaluate the vector line integral $\int_c (\nabla \cdot F)\,dr$, $F = xi + xyj + xy^2k$ and c is described by $x = 2t$, $y = t^2$, $z = 1 - t$ for t starting at $t = 0$ to $t = 1$.

② Evaluate the vector line integral $\int_c F \times dr$ when C represents $y = 4 - 4x$, $z = 2 - 2x$ from $(0,4,2)$ to $(1,0,0)$ and $F = (x - z)j$

Line Integrals 3)

A line integral in two dimensions may be written as

$$\int_C F(x,y)\,dw$$

There are three main features determining this integral:

$F(x,y)$: This is the scalar function to be integrated $F(x,y) = x^2 + 4y^2$.

C : This is the curve along which integration takes place. e.g $y = x^2$ or $x = \sin y$ or $x = t-1$, $y = t^2$ (where x, y are expressed in terms of parameter t).

dw : This gives the variable of the integration. Three main cases dx, dy, ds. s: is arc length and so indicates position along the curve C.

ds may be written as $ds = \sqrt{(dx)^2 + (dy)^2}$

or $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ ← قانون

...

A fourth case is when $F(x,y)\,dw$ has the form $F_1\,dx + F_2\,dy$. This is combination of the cases dx and dy.

The integral $\int_C F(x,y)\,ds$ represents the area beneath the surface $z = F(x,y)$ but above the curve C.

The integral $\int_C F(x,y)\,dx$ and $\int_C F(x,y)\,dy$ represent the projections of this area onto the xz and yz planes respectively.

A particular case of the integral $\int_C F(x,y)\,ds$ is the integral $\int_C 1\,ds$. This is a means of calculating the Length along a curve.

Example 1:-

Find $\int_C x(1+4y)\,dx$ where C is the curve $y = x^2$, starting from $x=0, y=0$ and ending at $x=1, y=1$.

Sol:-

$$\int_C x(1+4y)\,dx = \int_{x=0}^{1} x(1+4x^2)\,dx$$

$$\int (x+4xy)\,dx$$

$$\therefore \int_C (x + 4x(x^2))\,dx$$

$$= \int_{x=0}^{1} (x + 4x^3)\,dx$$

$$= \left[\frac{x^2}{2} + x^4 \right|_0^1 = \left(\frac{1}{2} + 1\right) - (0) = \frac{3}{2}$$

Example 2 :-

Find $\int_C x(1+4y)\,dy$ where C is the curve $y = x^2$ starting from $x=0, y=0$ and ending at $x=1, y=1$.

sol :-

$y = x^2$

$x = \mp\sqrt{y}$

$$\int_C x(1+4y)\,dy = \int_{y=0}^{1} y^{1/2}(1+4y)\,dy$$

$$= \left[\frac{2}{3} y^{3/2} + \frac{8}{5} y^{5/2} \right|_0^1 = \left(\frac{2}{3} + \frac{8}{5}\right) - 0$$

$$= \frac{34}{15}$$

Example 3 :-

Find $\int_C x(1+4y)\,ds$ where C is the curve $y = x^2$ starting from $x=0, y=0$ and ending at $x=1, y=1$. this is the same integral and curve as the previous two examples but the integration is now carried out with respect to s, The $\underbrace{\text{arc Length}}_{d\vec{s}}$ Parameter.

...

Sol:-

$$y = x^2, \quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1+(2x)^2}\, dx$$

$$= \sqrt{1+4x^2}\, dx$$

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$y = x^2$

$\frac{dy}{dx} = 2x$

so the integral is

$$\int_c x(1+4y)\,ds = \int_0^1 x(1+4x^2)\sqrt{1+4x^2}\,dx$$

$$= \int_{x=0}^{1} x(1+4x^2)^{3/2}\,dx \cdot \frac{8}{8}$$

$$= \frac{1}{8}\cdot\frac{2}{5}\left[1+4x^2\right]^{5/2}\Big|_0^1$$

$$= \frac{1}{20}\left[5^{5/2}-1\right] \simeq 2.745$$

Example 4:.

Find $\int_c xy\,dx$ where, on C, x and y are given in terms of aparamcter t by $x = 3t^2$, $y = t^3-1$ for t varying from 0 to 1

Sol:-

$$\int_c xy\,dx = \int_{t=0}^{1} 3t^2(t^3-1)\,6t\,dt$$

$x = 3t^2 \Longrightarrow \frac{dx}{dt} = 6t \Longrightarrow$

$dx = 6t\,dt$

$t =$
$$= \int_{0}^{1} (18t^6 - 18t^3)\,dt$$
$t =$

(51)

$$= \left[\frac{18}{7}t^7 - \frac{18}{4}t^4\right|_0^1 = \frac{18}{7} - \frac{9}{2} - 0 = \frac{-27}{14}$$

$$\int_c f(x,y)\,ds \neq \int_c f(x,y)\,dy \neq \int f(x,y)\,dx$$

H.W :- $F(x,y) = 2x + y^2$ find

① $\int_c F(x,y)\,dx$

② $\int_c F(x,y)\,dy$

③ $\int_c F(x,y)\,ds$ wher c is the Line $y = 2x$ from $(0,0)$ to $(1,2)$.

Example 5 :-

find $\int_c (2xy\,dx - 5x\,dy)$ where c is the curve $y = x^3$ $\quad 0 \leq x \leq 1$

Sol :- $y = x^3 \rightarrow \frac{dy}{dx} = 3x^2 \rightarrow dy = 3x^2\,dx$

$$\int_c (2xy\,dx - 5x\,dy = \int_0^1 (2x\,x^3 dx - 5x\,3x^2\,dx)$$

$$= \int_0^1 (2x^4 - 15x^3)\, dx$$

(52)

$$= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4 \right]_0^1 = \frac{2}{5} - \frac{15}{4} - 0 = \frac{67}{20}$$

Vector Line integrals

Vector → Diffe.
→ integ c

$$\int_c f(x,y,z)\,dr \quad \text{and} \quad \int F(x,y,z) \underline{\times\, dr}$$

where

$$dr = dx\,i + dy\,j + dz\,k$$

an integral of the form $\int_c f(x,y,z)\,dr$ becomes

$$\int_c f(x,y,z)\,dx\,i + \int_c f(x,y,z)\,dy\,j + \int_c f(x,y,z)\,dz\,k$$

Example1:-

Evaluate the integral $\int_c xy^2\,dr$ where c represents the contour $y = x^2$ from (0,0) to (1,1)

sol:-

$$\int_c xy^2\,dr = \int_c xy^2 (dx\,i + dy\,j)$$

$y = x^2$?

$y^2 = (x^2)$

$$= \int_c x y^2 \, dx \, i + \int_c x y^2 \, dy \, j$$

$$= \int_{x=0}^{1} x (x^2)^2 dx \, i + \int_{y=0}^{1} y^{1/2} y^2 \, dy \, j$$

$$= \int_0^1 x^5 \, dx \, i + \int_0^1 y^{5/2} \, dy \, j$$

$$= \frac{1}{6} x^6 \Big|_0^1 i + \frac{2}{7} y^{7/2} \Big|_0^1 j = \frac{1}{6} i + \frac{2}{7} j$$

(53)

$y = x^2$

$\sqrt{y} = x$

Example 2 :-

Find $I = \int_c x \, dr$ for contour c given parametrically by $x = \cos t$, $y = \sin t$, $z = t - \pi$ starting at $t = 0$ to $t = 2\pi$

dt

Sol :-

$x = \cos t$

$dx = -\sin t$

$z = t - \pi$

$dz = dt$

$$\int_c x \, dr = \int_c x (dx \, i + dy \, j + dz \, k)$$

$$= \int_0^{2\pi} \cos t \, (-\sin t \, dt \, i + \cos t \, dt \, j + dt \, k)$$

$$= -\int_0^{2\pi} \boxed{\cos t} \sin t \, dt \, i + \int_0^{2\pi} \cos^2 t \, dt \, j + \int_0^{2\pi} \cos t \, dt \, k$$

$$= -\frac{[\sin t]^2}{2} \Big|_0^{2\pi} i + \frac{1}{2} \int_0^{2\pi} (1 + \cos 2t) \, dt \, j + \sin t \Big|_0^{2\pi} k$$

$$= 0 i + \frac{1}{2} \left(t + \frac{1}{2} \sin 2t \right) \Big|_0^{2\pi} j + 0 k$$

$$= 0 i + \pi j + 0 k = \pi j$$

$\cos^2 \theta = \dfrac{1 + \cos 2\theta}{2}$

finition :- let V, W be a vector spaces.
function $T : V \to W$ is a Linear transforma
if

① $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in$
② $T(\lambda v) = \lambda T(v)$ for all $v \in V$ $\lambda \in F$

Example:- $T(x_1, x_2) = x_1 + x_2$ Linear Rule

T is a Linear transformation

① $T((x_1, x_2) + (y_1, y_2)) = T(x_1 + y_1, x_2 + y_2)$

$= x_1 + x_2 + y_1 + y_2$

$= T(x_1, x_2) + T(y_1, y_2)$

$x_1 + y_2 = 2$
$x_2 + y_2 = 3$
$x_1 + x_2$

② $T(\lambda(x_1, x_2)) = T(\lambda x_1, \lambda x_2)$

$= \lambda x_1 + \lambda x_2$

$= \lambda(x_1 + x_2)$

$= \lambda T(x_1, x_2)$

Example:- $T(x_1, x_2) = x_1 + x_2 + 1$ ✓

T is not Linear transformation

$T(2(1, 0)) = T(2, 0) = 3$

$2 + 0 + 1 = 3$

choosen

$2T(1,0) = 4 = 2[1+0+1]$

Properties of linear transformations

Let V and W be two vector space.

Suppose $T: V \to W$ is a linear transformation

Then

① $T(0) = 0$

② $T(-v) = -T(v)$ for all $v \in V$

③ $T(u-v) = T(u) - T(v)$ for all $u, v \in V$

④ if

$$v = c_1v_1 + c_2v_2 + c_3v_3 + \cdots c_nv_n$$

Then

$$T(v) = T(c_1v_1 + c_2v_2 + \cdots c_nv_n) =$$

$$c_1T(v_1) + c_2T(v_2) + c_3T(v_3) \cdots c_nT(v_n)$$

Linear transformation given by matrices

Suppose A is a matrix of size $m \times n$.

Given a vector

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \quad \text{define } T(v) = Av = A\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

ample :- $T: C^3 \longrightarrow C^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}$$

Linear

$x+y$ αx

T is Linear Transformation or not ?

sol:-

$$T(x+y) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(x_1+y_1) + (x_3+y_3) \\ -4(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} (2x_1 + x_3) + (2y_1 + y_3) \\ -4x_2 - 4y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_3 \\ -4y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

$$= T(x) + T(y)$$

d

$$T(\alpha x) = T\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right)$$ 57

$$= \begin{bmatrix} 2(\alpha x_1) + (\alpha x_3) \\ -4(\alpha x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(2x_1 + x_3) \\ \alpha(-4x_2) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}$$

$$= \alpha\, T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \alpha T(x)$$

T is Linear transformation

Example:- $S : C^3 \to C^3$, Show the non-linearity

input

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 2x_2 \\ 0 \\ x_1 + 3x_3 - 2 \end{bmatrix}$$

free

$4(1) + 2(2)$

$$3S\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 4+4 \\ 0 \\ 1+9-2 \end{bmatrix} = 3\begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 24 \end{bmatrix}$$

$x_1 + 3(3) - 2$

$$S\left(3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = S\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 24 \\ 0 \\ 28 \end{bmatrix} = \begin{bmatrix} 3(3) + 2(6) \\ 0 \\ 3 + 3(9) - 2 \end{bmatrix}$$

S is not Linear transformation