Chapter 1

Signals and Systems

1.1 INTRODUCTION

The concept and theory of signals and systems are needed in almost all electrical engineering fields and in many other engineering and scientific disciplines as well. In this chapter we introduce the mathematical description and representation of signals and systems and their classifications. We also define several important basic signals essential to our studies.

1.2 SIGNALS AND CLASSIFICATION OF SIGNALS

A *signal* is a function representing a physical quantity or variable, and typically it contains information about the behavior or nature of the phenomenon. For instance, in a *RC* circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable t. Usually t represents time. Thus, a signal is denoted by $x(t)$.

A. Continuous-Time and Discrete-Time Signals:

A signal $x(t)$ is a *continuous-time* signal if t is a continuous variable. If t is a discrete variable, that is, $x(t)$ is defined at discrete times, then $x(t)$ is a *discrete-time* signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a *sequence* of numbers, denoted by $\{x_n\}$ or $x[n]$, where n = integer. Illustrations of a continuous-time signal $x(t)$ and of a discrete-time signal $x[n]$ are shown in Fig. 1-1.

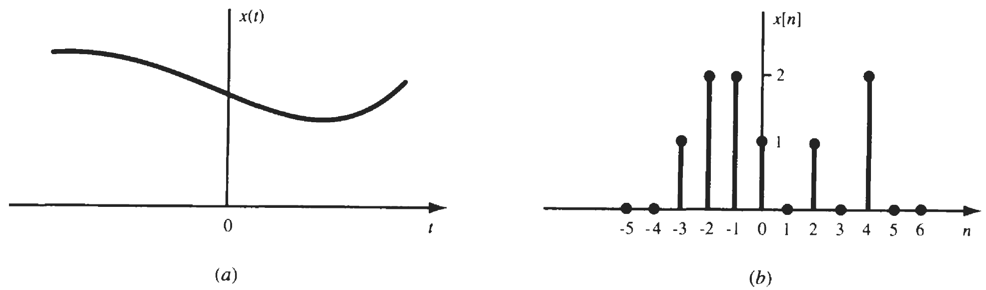

Fig. 1-1 Graphical representation of (a) continuous-time and (b) discrete-time signals.

A discrete-time signal $x[n]$ may represent a phenomenon for which the independent variable is inherently discrete. For instance, the daily closing stock market average is by its nature a signal that evolves at discrete points in time (that is, at the close of each day). On the other hand a discrete-time signal $x[n]$ may be obtained by *sampling* a continuous-time signal $x(t)$ such as

$$x(t_0),\ x(t_1), \ldots, x(t_n), \ldots$$

or in a shorter form as

$$x[0], x[1], \ldots, x[n], \ldots$$

or

$$x_0, x_1, \ldots, x_n, \ldots$$

where we understand that

$$x_n = x[n] = x(t_n)$$

and x_n's are called *samples* and the time interval between them is called the *sampling interval*. When the sampling intervals are equal (uniform sampling), then

$$x_n = x[n] = x(nT_s)$$

where the constant T_s is the sampling interval.

A discrete-time signal $x[n]$ can be defined in two ways:

1. We can specify a rule for calculating the nth value of the sequence. For example,

$$x[n] = x_n = \begin{cases} \left(\frac{1}{2}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

or

$$\{x_n\} = \left\{1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \left(\tfrac{1}{2}\right)^n, \ldots\right\}$$

2. We can also explicitly list the values of the sequence. For example, the sequence shown in Fig. 1-1(b) can be written as

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

or

$$\{x_n\} = \{1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2\}$$

We use the arrow to denote the $n = 0$ term. We shall use the convention that if no arrow is indicated, then the first term corresponds to $n = 0$ and all the values of the sequence are zero for $n < 0$.

The sum and product of two sequences are defined as follows:

$$\{c_n\} = \{a_n\} + \{b_n\} \longrightarrow c_n = a_n + b_n$$
$$\{c_n\} = \{a_n\}\{b_n\} \quad \longrightarrow c_n = a_n b_n$$
$$\{c_n\} = \alpha\{a_n\} \quad \longrightarrow c_n = \alpha a_n \qquad \alpha = \text{constant}$$

B. Analog and Digital Signals:

If a continuous-time signal $x(t)$ can take on any value in the continuous interval (a, b), where a may be $-\infty$ and b may be $+\infty$, then the continuous-time signal $x(t)$ is called an *analog* signal. If a discrete-time signal $x[n]$ can take on only a finite number of distinct values, then we call this signal a *digital* signal.

C. Real and Complex Signals:

A signal $x(t)$ is a *real* signal if its value is a real number, and a signal $x(t)$ is a *complex* signal if its value is a complex number. A general complex signal $x(t)$ is a function of the

form

$$x(t) = x_1(t) + jx_2(t) \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ are real signals and $j = \sqrt{-1}$.

Note that in Eq. (*1.1*) t represents either a continuous or a discrete variable.

D. Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modeled by a known function of time t. *Random* signals are those signals that take random values at any given time and must be characterized statistically. Random signals will not be discussed in this text.

E. Even and Odd Signals:

A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if

$$\begin{aligned} x(-t) &= x(t) \\ x[-n] &= x[n] \end{aligned} \tag{1.2}$$

A signal $x(t)$ or $x[n]$ is referred to as an *odd* signal if

$$\begin{aligned} x(-t) &= -x(t) \\ x[-n] &= -x[n] \end{aligned} \tag{1.3}$$

Examples of even and odd signals are shown in Fig. 1-2.

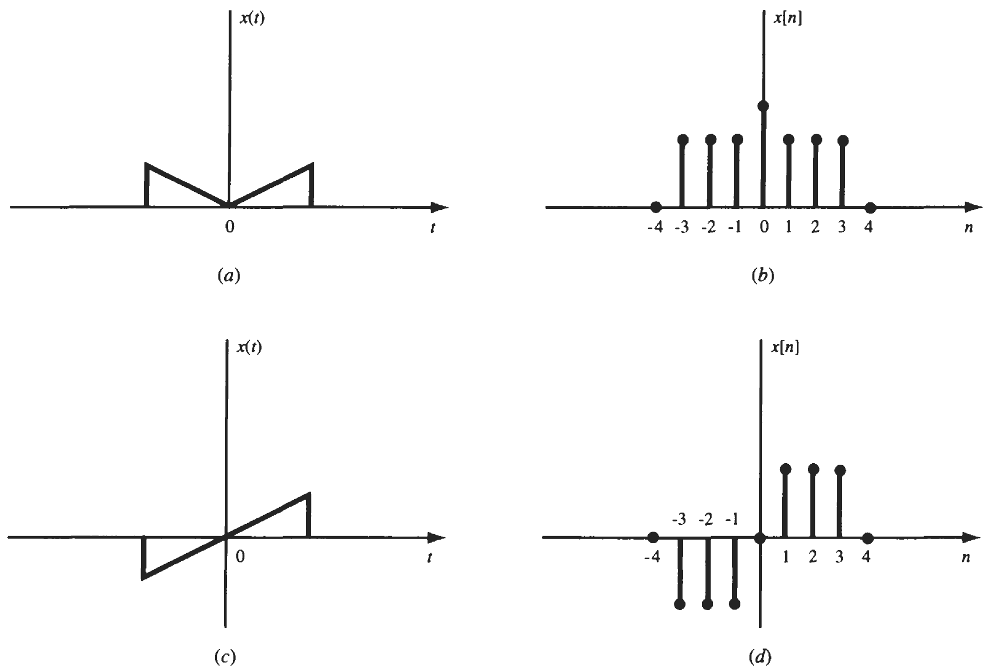

Fig. 1-2 Examples of even signals (a and b) and odd signals (c and d).

Any signal $x(t)$ or $x[n]$ can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_e(t) + x_o(t) \qquad x[n] = x_e[n] + x_o[n] \tag{1.4}$$

where

$$x_e(t) = \tfrac{1}{2}\{x(t) + x(-t)\} \quad \text{even part of } x(t) \qquad x_e[n] = \tfrac{1}{2}\{x[n] + x[-n]\} \quad \text{even part of } x[n] \tag{1.5}$$

$$x_o(t) = \tfrac{1}{2}\{x(t) - x(-t)\} \quad \text{odd part of } x(t) \qquad x_o[n] = \tfrac{1}{2}\{x[n] - x[-n]\} \quad \text{odd part of } x[n] \tag{1.6}$$

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal (Prob. 1.7).

F. Periodic and Nonperiodic Signals:

A continuous-time signal $x(t)$ is said to be *periodic with period* T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t \tag{1.7}$$

An example of such a signal is given in Fig. 1-3(a). From Eq. (1.7) or Fig. 1-3(a) it follows that

$$x(t+mT) = x(t) \tag{1.8}$$

for all t and any integer m. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which Eq. (1.7) holds. Note that this definition does not work for a constant

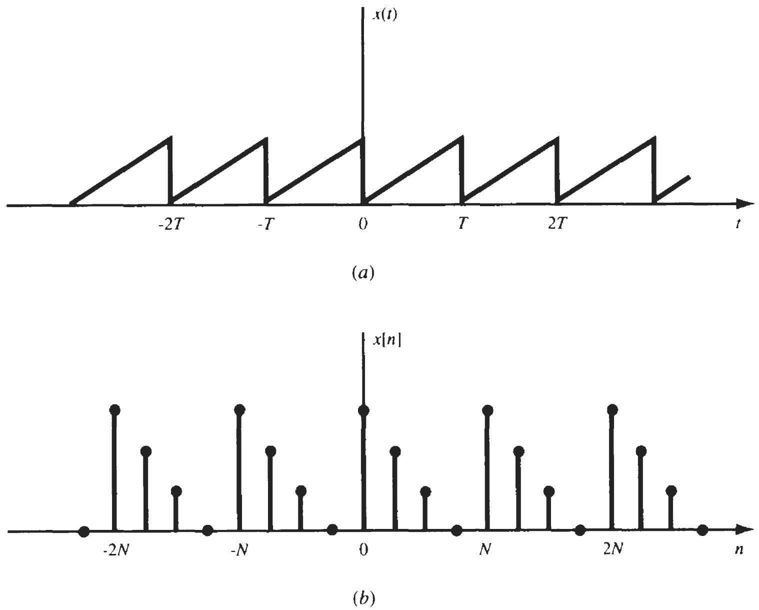

Fig. 1-3 Examples of periodic signals.

signal $x(t)$ (known as a dc signal). For a constant signal $x(t)$ the fundamental period is undefined since $x(t)$ is periodic for *any* choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a *nonperiodic* (or *aperiodic*) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) $x[n]$ is *periodic with period* N if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n \tag{1.9}$$

An example of such a sequence is given in Fig. 1-3(b). From Eq. (1.9) and Fig. 1-3(b) it follows that

$$x[n+mN] = x[n] \tag{1.10}$$

for all n and any integer m. The fundamental period N_0 of $x[n]$ is the smallest positive integer N for which Eq. (1.9) holds. Any sequence which is not periodic is called a *nonperiodic* (or *aperiodic*) sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic (Probs. 1.12 and 1.13). Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic (Probs. 1.14 and 1.15).

G. Energy and Power Signals:

Consider $v(t)$ to be the voltage across a resistor R producing a current $i(t)$. The instantaneous power $p(t)$ per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t) \tag{1.11}$$

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\, dt \quad \text{joules} \tag{1.12}$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} i^2(t)\, dt \quad \text{watts} \tag{1.13}$$

For an arbitrary continuous-time signal $x(t)$, the *normalized energy content* E of $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \tag{1.14}$$

The *normalized average power* P of $x(t)$ is defined as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \tag{1.15}$$

Similarly, for a discrete-time signal $x[n]$, the normalized energy content E of $x[n]$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{1.16}$$

The normalized average power P of $x[n]$ is defined as

$$P=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 \tag{1.17}$$

Based on definitions (*1.14*) to (*1.17*), the following classes of signals are defined:

1. $x(t)$ (or $x[n]$) is said to be an *energy* signal (or sequence) if and only if $0<E<\infty$, and so $P=0$.
2. $x(t)$ (or $x[n]$) is said to be a *power* signal (or sequence) if and only if $0<P<\infty$, thus implying that $E=\infty$.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period (Prob. 1.18).

1.3 BASIC CONTINUOUS-TIME SIGNALS

A. The Unit Step Function:

The *unit step* function $u(t)$, also known as the *Heaviside unit* function, is defined as

$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases} \tag{1.18}$$

which is shown in Fig. 1-4(*a*). Note that it is discontinuous at $t=0$ and that the value at $t=0$ is undefined. Similarly, the shifted unit step function $u(t-t_0)$ is defined as

$$u(t-t_0)=\begin{cases}1 & t>t_0\\ 0 & t<t_0\end{cases} \tag{1.19}$$

which is shown in Fig. 1-4(*b*).

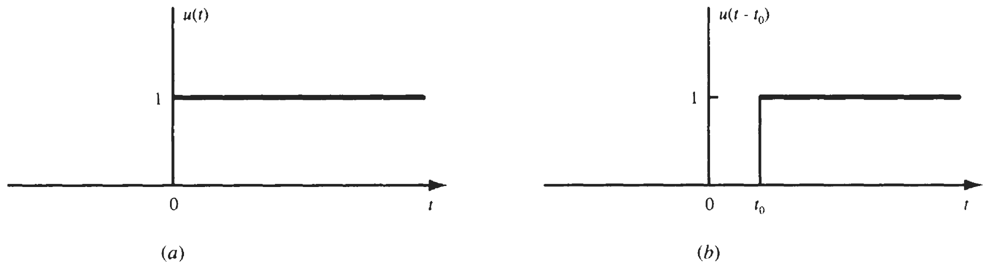

Fig. 1-4 (*a*) Unit step function; (*b*) shifted unit step function.

B. The Unit Impulse Function:

The *unit impulse* function $\delta(t)$, also known as the *Dirac delta* function, plays a central role in system analysis. Traditionally, $\delta(t)$ is often defined as the limit of a suitably chosen conventional function having unity area over an infinitesimal time interval as shown in

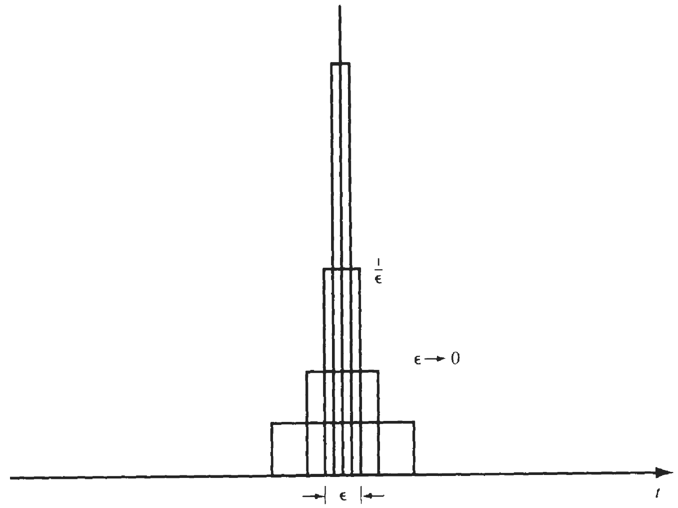

Fig. 1-5

Fig. 1-5 and possesses the following properties:

$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt=1$$

But an ordinary function which is everywhere 0 except at a single point must have the integral 0 (in the Riemann integral sense). Thus, $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$\int_{-\infty}^{\infty}\phi(t)\delta(t)\,dt=\phi(0) \tag{1.20}$$

where $\phi(t)$ is any regular function continuous at $t=0$.

An alternative definition of $\delta(t)$ is given by

$$\int_a^b\phi(t)\delta(t)\,dt=\begin{cases}\phi(0) & a<0<b\\ 0 & a<b<0 \quad\text{or}\quad 0<a<b\\ \text{undefined} & a=0 \quad\text{or}\quad b=0\end{cases} \tag{1.21}$$

Note that Eq. (*1.20*) or (*1.21*) is a symbolic expression and should not be considered an ordinary Riemann integral. In this sense, $\delta(t)$ is often called a *generalized function* and $\phi(t)$ is known as a *testing function*. A different class of testing functions will define a different generalized function (Prob. 1.24). Similarly, the delayed delta function $\delta(t-t_0)$ is defined by

$$\int_{-\infty}^{\infty}\phi(t)\delta(t-t_0)\,dt=\phi(t_0) \tag{1.22}$$

where $\phi(t)$ is any regular function continuous at $t=t_0$. For convenience, $\delta(t)$ and $\delta(t-t_0)$ are depicted graphically as shown in Fig. 1-6.

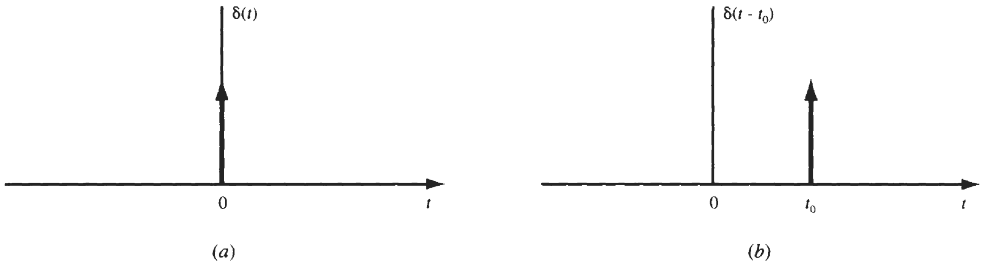

Fig. 1-6 (*a*) Unit impulse function; (*b*) shifted unit impulse function.

Some additional properties of $\delta(t)$ are

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{1.23}$$

$$\delta(-t) = \delta(t) \tag{1.24}$$

$$x(t)\delta(t) = x(0)\delta(t) \tag{1.25}$$

if $x(t)$ is continuous at $t = 0$.

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \tag{1.26}$$

if $x(t)$ is continuous at $t = t_0$.

Using Eqs. (*1.22*) and (*1.24*), any continuous-time signal $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \tag{1.27}$$

Generalized Derivatives:

If $g(t)$ is a generalized function, its nth generalized derivative $g^{(n)}(t) = d^n g(t)/dt^n$ is defined by the following relation:

$$\int_{-\infty}^{\infty} \phi(t) g^{(n)}(t)\,dt = (-1)^n \int_{-\infty}^{\infty} \phi^{(n)}(t) g(t)\,dt \tag{1.28}$$

where $\phi(t)$ is a testing function which can be differentiated an arbitrary number of times and vanishes outside some fixed interval and $\phi^{(n)}(t)$ is the nth derivative of $\phi(t)$. Thus, by Eqs. (*1.28*) and (*1.20*) the derivative of $\delta(t)$ can be defined as

$$\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt = -\phi'(0) \tag{1.29}$$

where $\phi(t)$ is a testing function which is continuous at $t = 0$ and vanishes outside some fixed interval and $\phi'(0) = d\phi(t)/dt|_{t=0}$. Using Eq. (*1.28*), the derivative of $u(t)$ can be shown to be $\delta(t)$ (Prob. 1.28); that is,

$$\delta(t) = u'(t) = \frac{du(t)}{dt} \tag{1.30}$$

Then the unit step function $u(t)$ can be expressed as

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \tag{1.31}$$

Note that the unit step function $u(t)$ is discontinuous at $t = 0$; therefore, the derivative of $u(t)$ as shown in Eq. (*1.30*) is not the derivative of a function in the ordinary sense and should be considered a generalized derivative in the sense of a generalized function. From Eq. (*1.31*) we see that $u(t)$ is undefined at $t = 0$ and

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

by Eq. (*1.21*) with $\phi(t) = 1$. This result is consistent with the definition (*1.18*) of $u(t)$.

C. Complex Exponential Signals:

The *complex exponential* signal

$$x(t) = e^{j\omega_0 t} \tag{1.32}$$

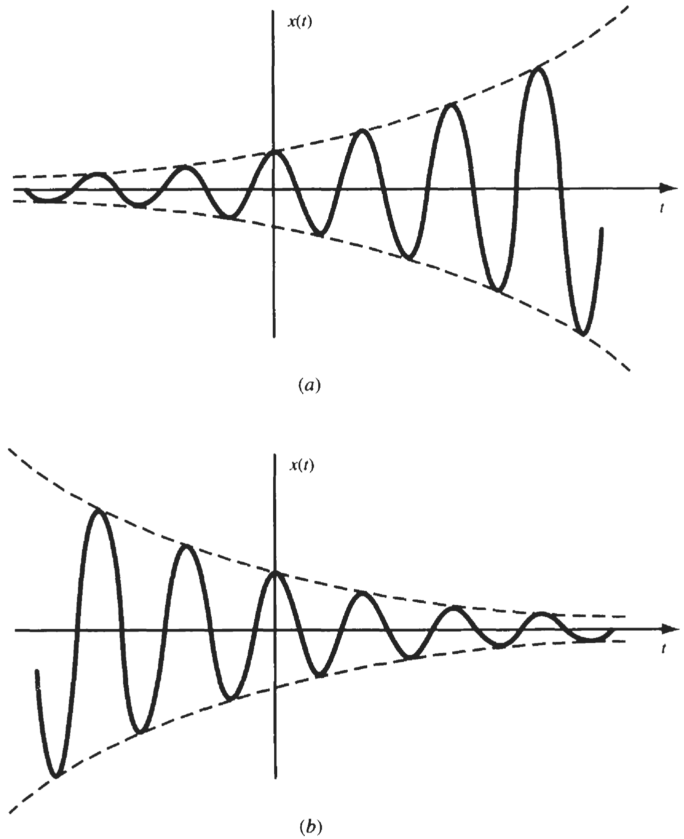

Fig. 1-7 (*a*) Exponentially increasing sinusoidal signal; (*b*) exponentially decreasing sinusoidal signal.

is an important example of a complex signal. Using Euler's formula, this signal can be defined as

$$x(t) = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \tag{1.33}$$

Thus, $x(t)$ is a complex signal whose real part is $\cos \omega_0 t$ and imaginary part is $\sin \omega_0 t$. An important property of the complex exponential signal $x(t)$ in Eq. (*1.32*) is that it is periodic. The fundamental period T_0 of $x(t)$ is given by (Prob. 1.9)

$$T_0 = \frac{2\pi}{\omega_0} \tag{1.34}$$

Note that $x(t)$ is periodic for any value of ω_0.

General Complex Exponential Signals:

Let $s = \sigma + j\omega$ be a complex number. We define $x(t)$ as

$$x(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t}(\cos \omega t + j \sin \omega t) \tag{1.35}$$

Then signal $x(t)$ in Eq. (*1.35*) is known as a *general complex exponential* signal whose real part $e^{\sigma t} \cos \omega t$ and imaginary part $e^{\sigma t} \sin \omega t$ are exponentially increasing ($\sigma > 0$) or decreasing ($\sigma < 0$) sinusoidal signals (Fig. 1-7).

Real Exponential Signals:

Note that if $s = \sigma$ (a real number), then Eq. (*1.35*) reduces to a *real exponential* signal

$$x(t) = e^{\sigma t} \tag{1.36}$$

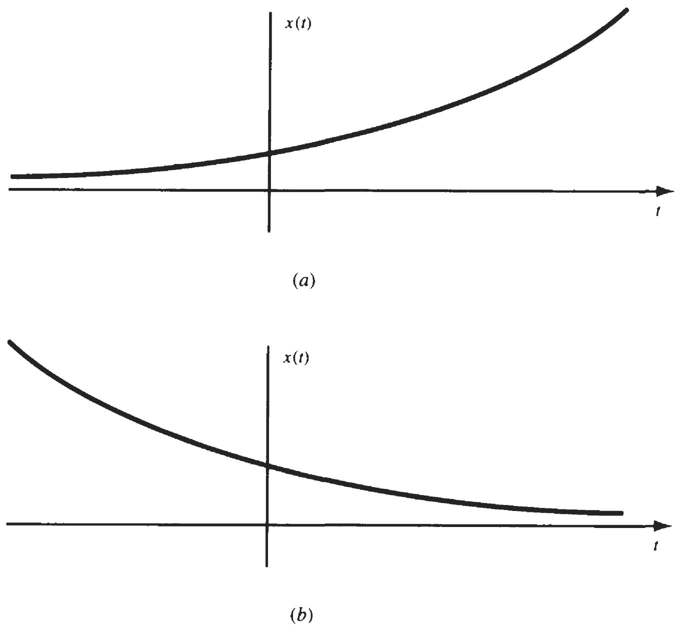

Fig. 1-8 Continuous-time real exponential signals. (*a*) $\sigma > 0$; (*b*) $\sigma < 0$.

As illustrated in Fig. 1-8, if $\sigma > 0$, then $x(t)$ is a growing exponential; and if $\sigma < 0$, then $x(t)$ is a decaying exponential.

D. Sinusoidal Signals:

A continuous-time *sinusoidal* signal can be expressed as

$$x(t) = A \cos(\omega_0 t + \theta) \tag{1.37}$$

where A is the *amplitude* (real), ω_0 is the *radian frequency* in radians per second, and θ is the *phase angle* in radians. The sinusoidal signal $x(t)$ is shown in Fig. 1-9, and it is periodic with fundamental period

$$T_0 = \frac{2\pi}{\omega_0} \tag{1.38}$$

The reciprocal of the fundamental period T_0 is called the *fundamental frequency* f_0:

$$f_0 = \frac{1}{T_0} \quad \text{hertz (Hz)} \tag{1.39}$$

From Eqs. (*1.38*) and (*1.39*) we have

$$\omega_0 = 2\pi f_0 \tag{1.40}$$

which is called the *fundamental angular frequency*. Using Euler's formula, the sinusoidal signal in Eq. (*1.37*) can be expressed as

$$A \cos(\omega_0 t + \theta) = A \operatorname{Re}\{e^{j(\omega_0 t + \theta)}\} \tag{1.41}$$

where "Re" denotes "real part of." We also use the notation "Im" to denote "imaginary part of." Then

$$A \operatorname{Im}\{e^{j(\omega_0 t + \theta)}\} = A \sin(\omega_0 t + \theta) \tag{1.42}$$

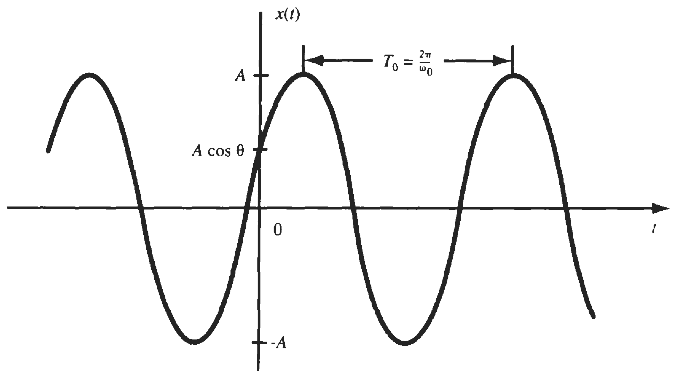

Fig. 1-9 Continuous-time sinusoidal signal.

1.4 BASIC DISCRETE-TIME SIGNALS

A. The Unit Step Sequence:

The *unit step* sequence $u[n]$ is defined as

$$u[n]=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases} \tag{1.43}$$

which is shown in Fig. 1-10(*a*). Note that the value of $u[n]$ at $n=0$ is defined [unlike the continuous-time step function $u(t)$ at $t=0$] and equals unity. Similarly, the shifted unit step sequence $u[n-k]$ is defined as

$$u[n-k]=\begin{cases}1 & n\geq k\\ 0 & n<k\end{cases} \tag{1.44}$$

which is shown in Fig. 1-10(*b*).

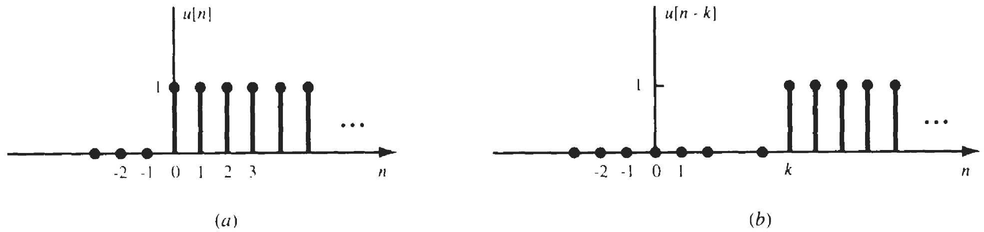

Fig. 1-10 (*a*) Unit step sequence; (*b*) shifted unit step sequence.

B. The Unit Impulse Sequence:

The *unit impulse* (or *unit sample*) sequence $\delta[n]$ is defined as

$$\delta[n]=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases} \tag{1.45}$$

which is shown in Fig. 1-11(*a*). Similarly, the shifted unit impulse (or sample) sequence $\delta[n-k]$ is defined as

$$\delta[n-k]=\begin{cases}1 & n=k\\ 0 & n\neq k\end{cases} \tag{1.46}$$

which is shown in Fig. 1-11(*b*).

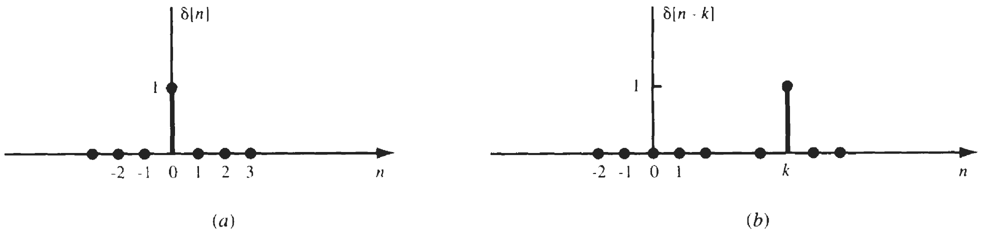

Fig. 1-11 (*a*) Unit impulse (sample) sequence; (*b*) shifted unit impulse sequence.

Unlike the continuous-time unit impulse function $\delta(t)$, $\delta[n]$ is defined without mathematical complication or difficulty. From definitions (*1.45*) and (*1.46*) it is readily seen that

$$x[n]\delta[n]=x[0]\delta[n] \tag{1.47}$$

$$x[n]\delta[n-k]=x[k]\delta[n-k] \tag{1.48}$$

which are the discrete-time counterparts of Eqs. (*1.25*) and (*1.26*), respectively. From definitions (*1.43*) to (*1.46*), $\delta[n]$ and $u[n]$ are related by

$$\delta[n]=u[n]-u[n-1] \tag{1.49}$$

$$u[n]=\sum_{k=-\infty}^{n}\delta[k] \tag{1.50}$$

which are the discrete-time counterparts of Eqs. (*1.30*) and (*1.31*), respectively.

Using definition (*1.46*), any sequence $x[n]$ can be expressed as

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \tag{1.51}$$

which corresponds to Eq. (*1.27*) in the continuous-time signal case.

C. Complex Exponential Sequences:

The *complex exponential* sequence is of the form

$$x[n]=e^{j\Omega_0 n} \tag{1.52}$$

Again, using Euler's formula, $x[n]$ can be expressed as

$$x[n]=e^{j\Omega_0 n}=\cos\Omega_0 n+j\sin\Omega_0 n \tag{1.53}$$

Thus $x[n]$ is a complex sequence whose real part is $\cos\Omega_0 n$ and imaginary part is $\sin\Omega_0 n$.

Periodicity of $e^{j\Omega_0 n}$:

In order for $e^{j\Omega_0 n}$ to be periodic with period N (>0), Ω_0 must satisfy the following condition (Prob. 1.11):

$$\frac{\Omega_0}{2\pi}=\frac{m}{N}\qquad m=\text{positive integer} \tag{1.54}$$

Thus the sequence $e^{j\Omega_0 n}$ is not periodic for any value of Ω_0. It is periodic only if $\Omega_0/2\pi$ is a rational number. Note that this property is quite different from the property that the continuous-time signal $e^{j\omega_0 t}$ is periodic for any value of ω_0. Thus, if Ω_0 satisfies the periodicity condition in Eq. (*1.54*), $\Omega_0\neq 0$, and N and m have no factors in common, then the fundamental period of the sequence $x[n]$ in Eq. (*1.52*) is N_0 given by

$$N_0=m\left(\frac{2\pi}{\Omega_0}\right) \tag{1.55}$$

Another very important distinction between the discrete-time and continuous-time complex exponentials is that the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0 but that this is not the case for the signals $e^{j\Omega_0 n}$.

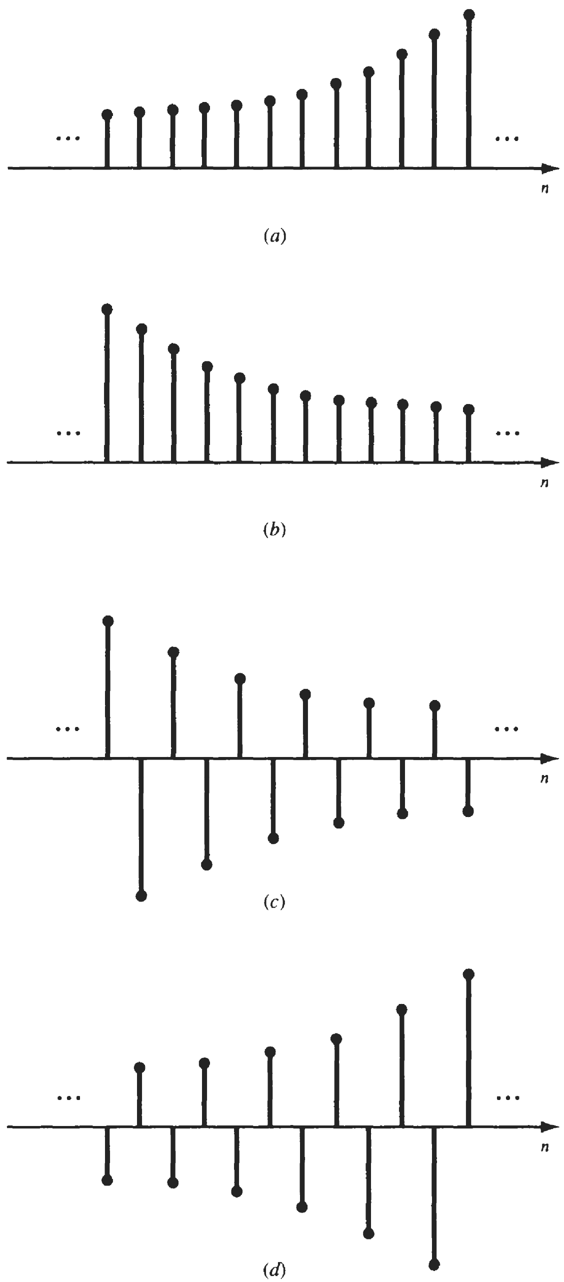

Fig. 1-12 Real exponential sequences. $(a)\ \alpha > 1$; $(b)\ 1 > \alpha > 0$; $(c)\ 0 > \alpha > -1$; $(d)\ \alpha < -1$.

Consider the complex exponential sequence with frequency $(\Omega_0 + 2\pi k)$, where k is an integer:

$$e^{j(\Omega_0+2\pi k)n} = e^{j\Omega_0 n}e^{j2\pi kn} = e^{j\Omega_0 n} \tag{1.56}$$

since $e^{j2\pi kn} = 1$. From Eq. (*1.56*) we see that the complex exponential sequence at frequency Ω_0 is the same as that at frequencies $(\Omega_0 \pm 2\pi)$, $(\Omega_0 \pm 4\pi)$, and so on. Therefore, in dealing with discrete-time exponentials, we need only consider an interval of length 2π in which to choose Ω_0. Usually, we will use the interval $0 \le \Omega_0 < 2\pi$ or the interval $-\pi \le \Omega_0 < \pi$.

General Complex Exponential Sequences:

The most general complex exponential sequence is often defined as

$$x[n] = C\alpha^n \tag{1.57}$$

where C and α are in general complex numbers. Note that Eq. (*1.52*) is the special case of Eq. (*1.57*) with $C = 1$ and $\alpha = e^{j\Omega_0}$.

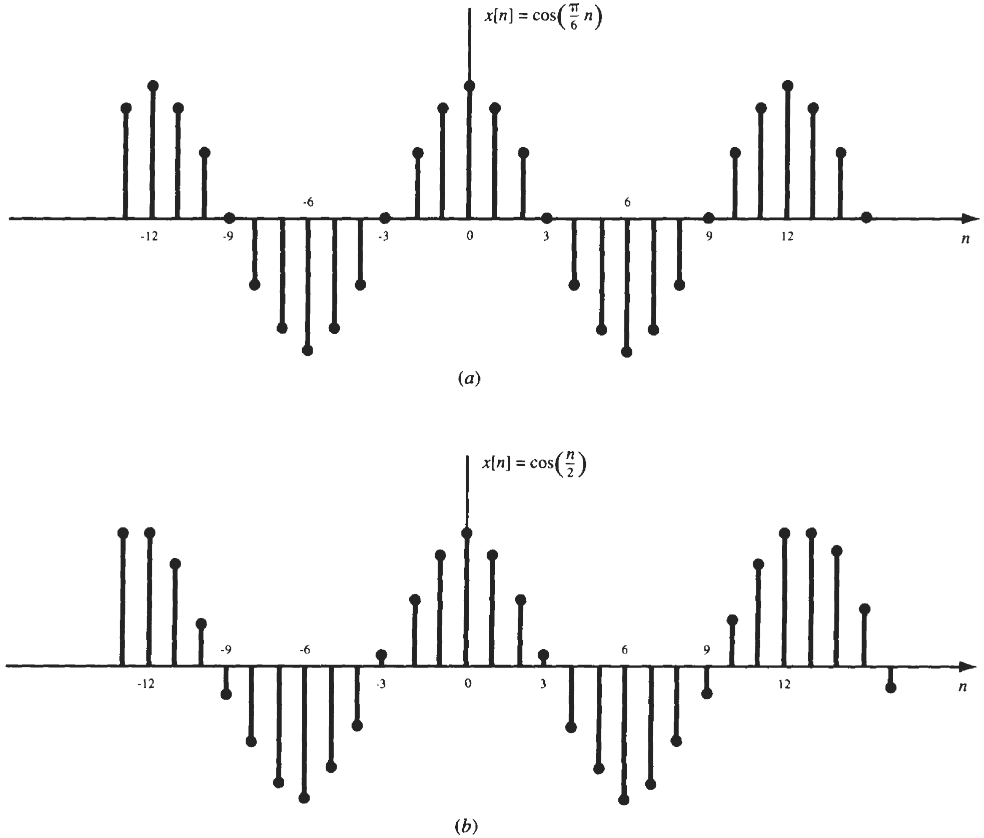

Fig. 1-13 Sinusoidal sequences. $(a)\ x[n] = \cos(\pi n/6)$; $(b)\ x[n] = \cos(n/2)$.

Real Exponential Sequences:

If C and α in Eq. (*1.57*) are both real, then $x[n]$ is a real exponential sequence. Four distinct cases can be identified: $\alpha > 1$, $0 < \alpha < 1$, $-1 < \alpha < 0$, and $\alpha < -1$. These four real exponential sequences are shown in Fig. 1-12. Note that if $\alpha = 1$, $x[n]$ is a constant sequence, whereas if $\alpha = -1$, $x[n]$ alternates in value between $+C$ and $-C$.

D. Sinusoidal Sequences:

A *sinusoidal* sequence can be expressed as

$$x[n] = A\cos(\Omega_0 n + \theta) \tag{1.58}$$

If n is dimensionless, then both Ω_0 and θ have units of radians. Two examples of sinusoidal sequences are shown in Fig. 1-13. As before, the sinusoidal sequence in Eq. (*1.58*) can be expressed as

$$A\cos(\Omega_0 n + \theta) = A\,\text{Re}\{e^{j(\Omega_0 n + \theta)}\} \tag{1.59}$$

As we observed in the case of the complex exponential sequence in Eq. (*1.52*), the same observations [Eqs. (*1.54*) and (*1.56*)] also hold for sinusoidal sequences. For instance, the sequence in Fig. 1-13(*a*) is periodic with fundamental period 12, but the sequence in Fig. 1-13(*b*) is not periodic.

1.5 SYSTEMS AND CLASSIFICATION OF SYSTEMS

A. System Representation:

A *system* is a mathematical model of a physical process that relates the *input* (or *excitation*) signal to the *output* (or *response*) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a *transformation* (or *mapping*) of x into y. This transformation is represented by the mathematical notation

$$y = \mathbf{T}x \tag{1.60}$$

where $\mathbf{T}$ is the *operator* representing some well-defined rule by which x is transformed into y. Relationship (*1.60*) is depicted as shown in Fig. 1-14(*a*). Multiple input and/or output signals are possible as shown in Fig. 1-14(*b*). We will restrict our attention for the most part in this text to the single-input, single-output case.

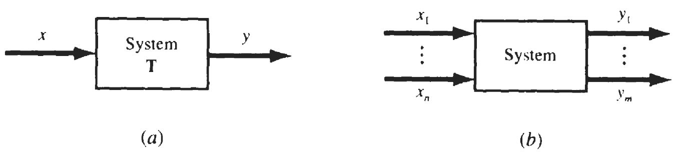

Fig. 1-14 System with single or multiple input and output signals.

B. Continuous-Time and Discrete-Time Systems:

If the input and output signals x and y are continuous-time signals, then the system is called a *continuous-time system* [Fig. 1-15(*a*)]. If the input and output signals are discrete-time signals or sequences, then the system is called a *discrete-time system* [Fig. 1-15(*b*)].

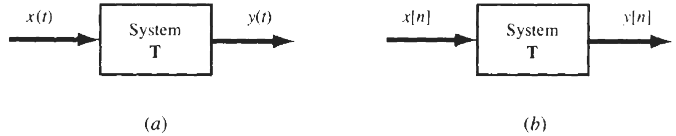

Fig. 1-15 (*a*) Continuous-time system; (*b*) discrete-time system.

C. Systems with Memory and without Memory

A system is said to be *memoryless* if the output at any time depends on only the input at that same time. Otherwise, the system is said to have *memory*. An example of a memoryless system is a resistor R with the input $x(t)$ taken as the current and the voltage taken as the output $y(t)$. The input-output relationship (Ohm's law) of a resistor is

$$y(t) = Rx(t) \tag{1.61}$$

An example of a system with memory is a capacitor C with the current as the input $x(t)$ and the voltage as the output $y(t)$; then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau \tag{1.62}$$

A second example of a system with memory is a discrete-time system whose input and output sequences are related by

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{1.63}$$

D. Causal and Noncausal Systems:

A system is called *causal* if its output $y(t)$ at an arbitrary time $t = t_0$ depends on only the input $x(t)$ for $t \le t_0$. That is, the output of a causal system at the present time depends on only the present and/or past values of the input, not on its future values. Thus, in a causal system, it is not possible to obtain an output before an input is applied to the system. A system is called *noncausal* if it is not causal. Examples of noncausal systems are

$$y(t) = x(t+1) \tag{1.64}$$

$$y[n] = x[-n] \tag{1.65}$$

Note that all memoryless systems are causal, but not vice versa.

E. Linear Systems and Nonlinear Systems:

If the operator **T** in Eq. (*1.60*) satisfies the following two conditions, then **T** is called a *linear operator* and the system represented by a linear operator **T** is called a *linear system*:

1. ***Additivity:***

Given that $\mathbf{T}x_1 = y_1$ and $\mathbf{T}x_2 = y_2$, then

$$\mathbf{T}\{x_1 + x_2\} = y_1 + y_2 \tag{1.66}$$

for any signals x_1 and x_2.

2. ***Homogeneity*** (or ***Scaling***):

$$\mathbf{T}\{\alpha x\} = \alpha y \tag{1.67}$$

for any signals x and any scalar α.

Any system that does not satisfy Eq. (*1.66*) and/or Eq. (*1.67*) is classified as a *nonlinear* system. Equations (*1.66*) and (*1.67*) can be combined into a single condition as

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2 \tag{1.68}$$

where α_1 and α_2 are arbitrary scalars. Equation (*1.68*) is known as the *superposition property*. Examples of linear systems are the resistor [Eq. (*1.61*)] and the capacitor [Eq. (*1.62*)]. Examples of nonlinear systems are

$$y = x^2 \tag{1.69}$$

$$y = \cos x \tag{1.70}$$

Note that a consequence of the homogeneity (or scaling) property [Eq. (*1.67*)] of linear systems is that *a zero input yields a zero output*. This follows readily by setting $\alpha = 0$ in Eq. (*1.67*). This is another important property of linear systems.

F. Time-Invariant and Time-Varying Systems:

A system is called *time-invariant* if a time shift (delay or advance) in the input signal causes the same time shift in the output signal. Thus, for a continuous-time system, the system is time-invariant if

$$\mathbf{T}\{x(t-\tau)\} = y(t-\tau) \tag{1.71}$$

for any real value of τ. For a discrete-time system, the system is time-invariant (or *shift-invariant*) if

$$\mathbf{T}\{x[n-k]\} = y[n-k] \tag{1.72}$$

for any integer k. A system which does not satisfy Eq. (*1.71*) (continuous-time system) or Eq. (*1.72*) (discrete-time system) is called a *time-varying* system. To check a system for time-invariance, we can compare the shifted output with the output produced by the shifted input (Probs. 1.33 to 1.39).

G. Linear Time-Invariant Systems

If the system is linear and also time-invariant, then it is called a *linear time-invariant* (LTI) system.

H. Stable Systems:

A system is *bounded-input/bounded-output* (BIBO) *stable* if for any bounded input x defined by

$$|x| \le k_1 \tag{1.73}$$

the corresponding output y is also bounded defined by

$$|y| \le k_2 \tag{1.74}$$

where k_1 and k_2 are finite real constants. Note that there are many other definitions of stability. (See Chap. 7.)

I. Feedback Systems:

A special class of systems of great importance consists of systems having *feedback*. In a *feedback system*, the output signal is fed back and added to the input to the system as shown in Fig. 1-16.

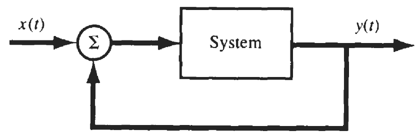

Fig. 1-16 Feedback system.

Solved Problems

SIGNALS AND CLASSIFICATION OF SIGNALS

1.1. A continuous-time signal $x(t)$ is shown in Fig. 1-17. Sketch and label each of the following signals.

(*a*) $x(t-2)$; (*b*) $x(2t)$; (*c*) $x(t/2)$; (*d*) $x(-t)$

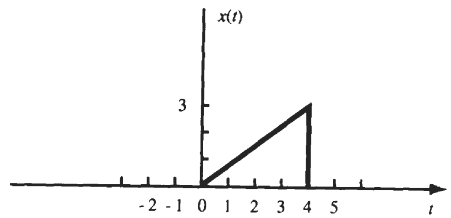

Fig. 1-17

(*a*) $x(t-2)$ is sketched in Fig. 1-18(*a*).
(*b*) $x(2t)$ is sketched in Fig. 1-18(*b*).
(*c*) $x(t/2)$ is sketched in Fig. 1-18(*c*).
(*d*) $x(-t)$ is sketched in Fig. 1-18(*d*).

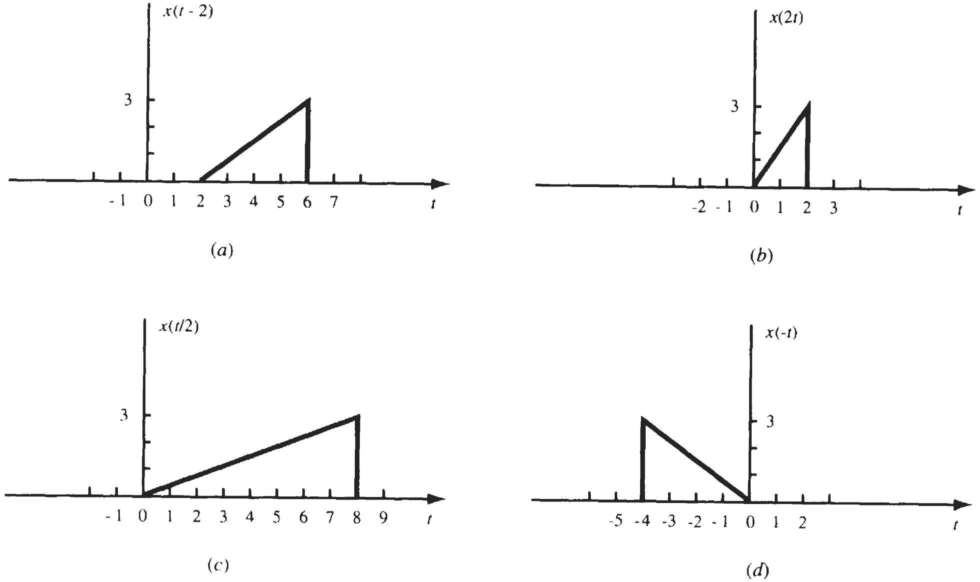

Fig. 1-18

1.2. A discrete-time signal $x[n]$ is shown in Fig. 1-19. Sketch and label each of the following signals.

(*a*) $x[n-2]$; (*b*) $x[2n]$; (*c*) $x[-n]$; (*d*) $x[-n+2]$

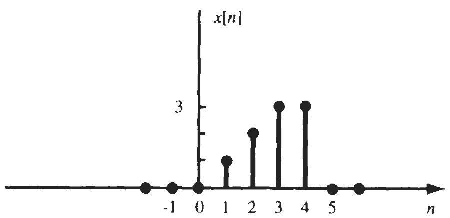

Fig. 1-19

(*a*) $x[n-2]$ is sketched in Fig. 1-20(*a*).
(*b*) $x[2n]$ is sketched in Fig. 1-20(*b*).
(*c*) $x[-n]$ is sketched in Fig. 1-20(*c*).
(*d*) $x[-n+2]$ is sketched in Fig. 1-20(*d*).

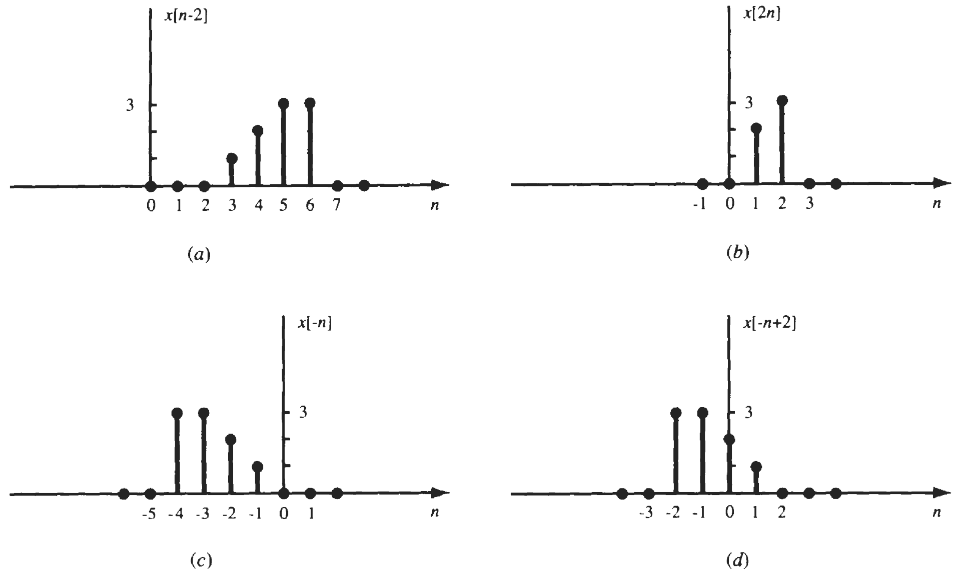

Fig. 1-20

1.3. Given the continuous-time signal specified by

$$x(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

determine the resultant discrete-time sequence obtained by uniform sampling of $x(t)$ with a sampling interval of (*a*) 0.25 s, (*b*) 0.5 s, and (*c*) 1.0 s.

It is easier to take the graphical approach for this problem. The signal $x(t)$ is plotted in Fig. 1-21(*a*). Figures 1-21(*b*) to (*d*) give plots of the resultant sampled sequences obtained for the three specified sampling intervals.

(*a*) $T_s = 0.25$ s. From Fig. 1-21(*b*) we obtain

$$x[n] = \{\ldots, 0, 0.25, 0.5, 0.75, \underset{\uparrow}{1}, 0.75, 0.5, 0.25, 0, \ldots\}$$

(*b*) $T_s = 0.5$ s. From Fig. 1-21(*c*) we obtain

$$x[n] = \{\ldots, 0, 0.5, \underset{\uparrow}{1}, 0.5, 0, \ldots\}$$

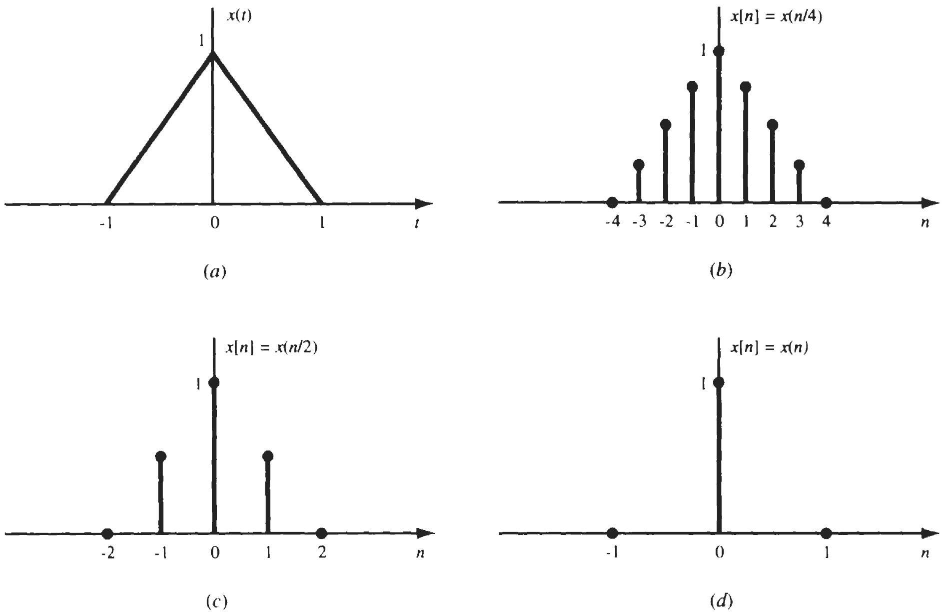

Fig. 1-21

(c) $T_s = 1$ s. From Fig. 1-21(d) we obtain

$$x[n] = \{\ldots, 0, \underset{\uparrow}{1}, 0, \ldots\} = \delta[n]$$

1.4. Using the discrete-time signals $x_1[n]$ and $x_2[n]$ shown in Fig. 1-22, represent each of the following signals by a graph and by a sequence of numbers.

(a) $y_1[n] = x_1[n] + x_2[n]$; (b) $y_2[n] = 2x_1[n]$; (c) $y_3[n] = x_1[n]x_2[n]$

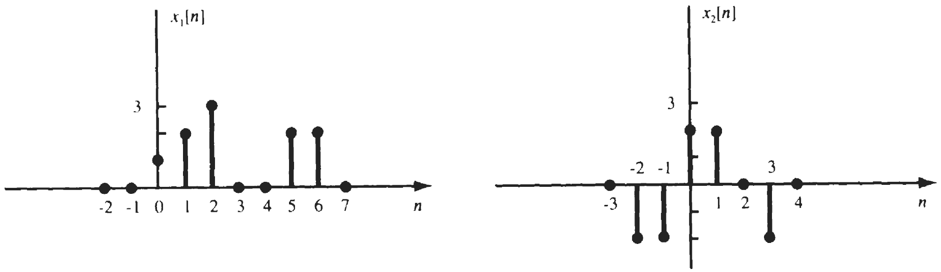

Fig. 1-22

(a) $y_1[n]$ is sketched in Fig. 1-23(a). From Fig. 1-23(a) we obtain

$$y_1[n] = \{\ldots, 0, -2, -2, \underset{\uparrow}{3}, 4, 3, -2, 0, 2, 2, 0, \ldots\}$$

(b) $y_2[n]$ is sketched in Fig. 1-23(b). From Fig. 1-23(b) we obtain

$$y_2[n] = \{\ldots, 0, \underset{\uparrow}{2}, 4, 6, 0, 0, 4, 4, 0, \ldots\}$$

(c) $y_3[n]$ is sketched in Fig. 1-23(c). From Fig. 1-23(c) we obtain

$$y_3[n] = \{\ldots, 0, \underset{\uparrow}{2}, 4, 0, \ldots\}$$

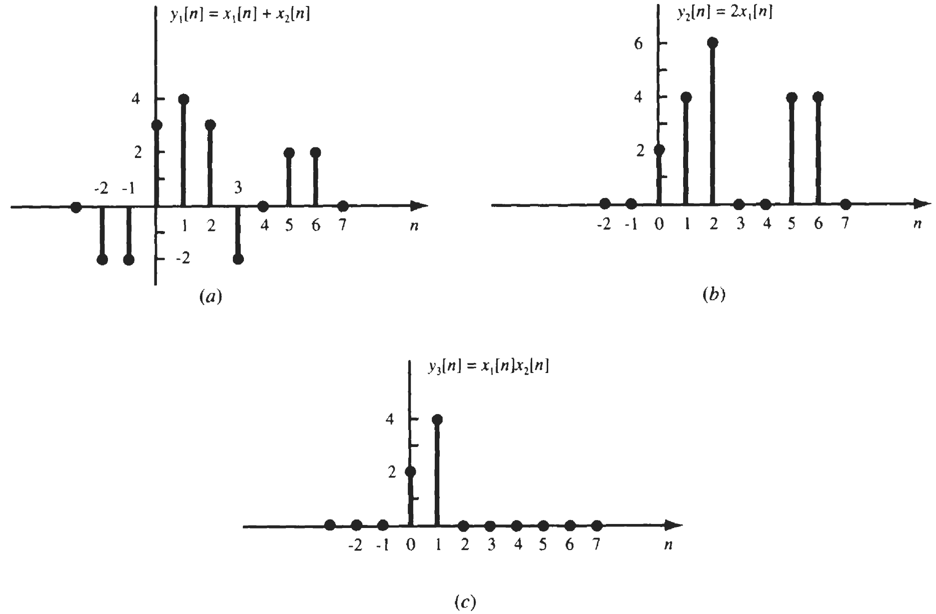

Fig. 1-23

1.5. Sketch and label the even and odd components of the signals shown in Fig. 1-24.

Using Eqs. (1.5) and (1.6), the even and odd components of the signals shown in Fig. 1-24 are sketched in Fig. 1-25.

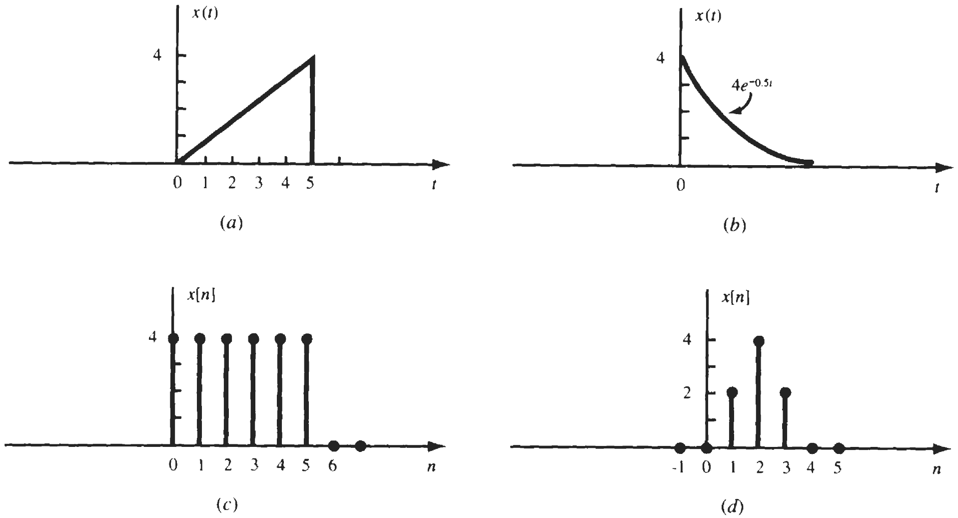

Fig. 1-24

1.6. Find the even and odd components of $x(t) = e^{jt}$.

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of e^{jt}, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From Eqs. (*1.5*) and (*1.6*) and using Euler's formula, we obtain

$$x_e(t) = \tfrac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \tfrac{1}{2}(e^{jt} - e^{-jt}) = j \sin t$$

1.7. Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signal is an odd signal.

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ is even and $x_2(t)$ is odd, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

and $x(t)$ is odd. Note that in the above proof, variable t represents either a continuous or a discrete variable.

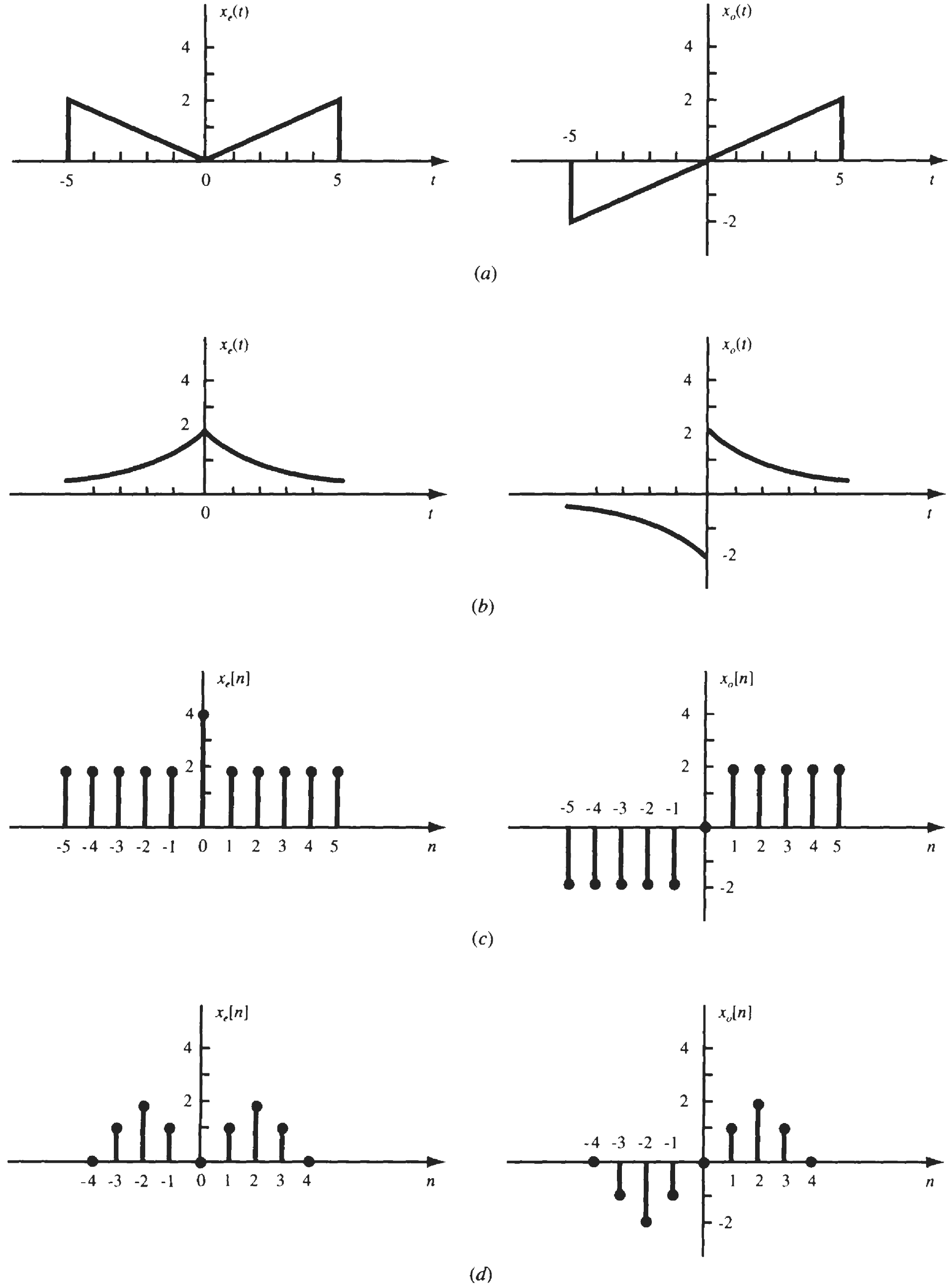

Fig. 1-25